Computer and Robot Vision I

Chapter 5 Mathematical Morphology

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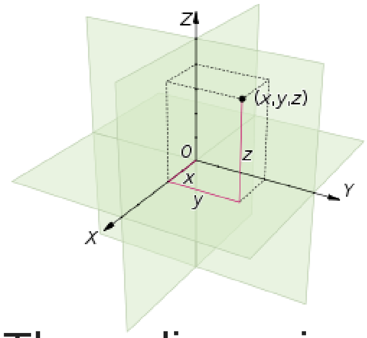
5.1 Introduction

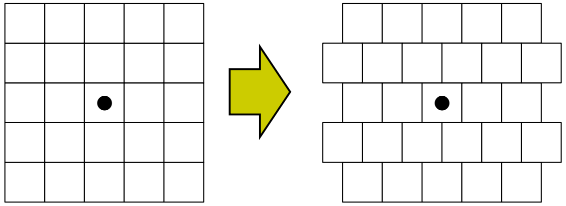
- mathematical morphology works on shape
- shape: prime carrier of information in machine vision
- morphological operations: simplify image data, preserve essential shape characteristics, eliminate irrelevancies
- shape: correlates directly with decomposition of object, object features, object surface defects, assembly defects

5.2 Binary Morphology

- set theory: language of binary mathematical morphology
- sets in mathematical morphology: represent shapes
- Euclidean N-space: E^N
- Discrete Euclidean N-space: Z^N
- N=2: hexagonal grid, square grid



Three-dimensional Euclidean space



5.2 Binary Morphology (cont')

- dilation, erosion: primary morphological operations
- opening, closing: composed from dilation, erosion
- opening, closing: related to shape representation, decomposition, primitive extraction

5.2.1 Binary Dilation

- dilation: combines two sets by vector addition of set elements
- dilation of *A* by *B*: $A \oplus B$

$$A \oplus B = \{c \in E^N \mid c = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

- addition commutative ➔ dilation commutative:

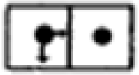
$$A \oplus B = B \oplus A$$

- binary dilation: Minkowski addition

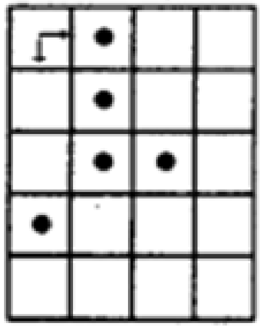
5.2.1 Binary Dilation (cont')

$$A = \{(0,1),(1,1),(2,1),(2,2),(3,0)\}$$

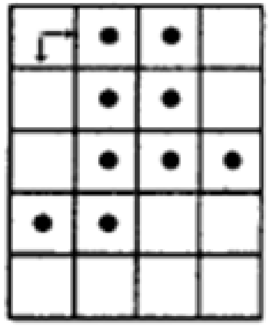
$$B = \{(0,0),(0,1)\}$$



B



A



$A \oplus B$

$$A \oplus B = \{(0,1),\ (1,1),\ (2,1),\ (3,0)\ (0,2),\ (1,2),\ (2,2),\ (2,3),\ (3,1)\}$$

Figure 5.1 Dilation operation represented in terms of sets, and their corresponding binary images.

5.2.1 Binary Dilation (cont')

- *A*: referred as set: image
- *B*: structuring element: kernel
- dilation by disk: isotropic swelling or expansion

5.2.1 Binary Dilation (cont')

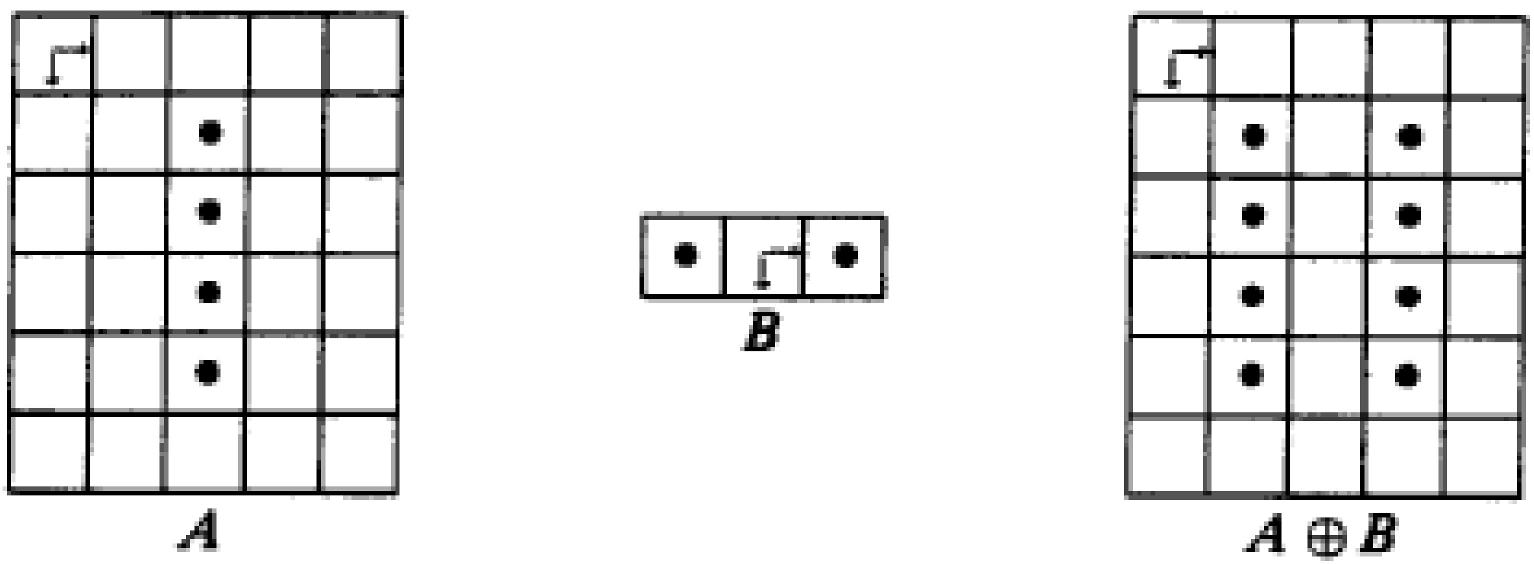


Figure 5.2 Set A dilated by a structuring element B that does not contain the origin. As a result, the dilated set is not even guaranteed to have a single point in common with A. However, there are always translations of $A \oplus B$ that can contain A.

5.2.1 Binary Dilation (cont')

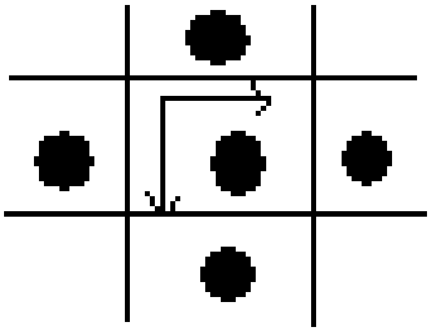
- dilation by kernel without origin: might not have common pixels with *A*
- translation of dilation: always can contain *A*

5.2.1 Binary Dilation (cont')

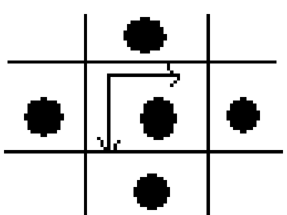
- =lena.bin.128=

5.2.1 Binary Dilation (cont')

- =lena.bin.dil=
- By structuring element :

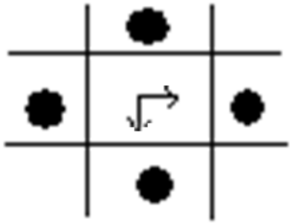


5.2.1 Binary Dilation (cont')



5.2.1 Binary Dilation (cont')

$$J = I \cap (I \oplus N_4) \text{ for noise removal}$$



- N_4: set of four 4-neighbors of (0, 0) but not (0, 0)
- 4-isolated pixels removed
- only points in I with at least one of its 4-neighbors remain
- A_t: translation of A by the point t

$$A_t = \{c \in E^N \mid c = a + t \text{ for some } a \in A\}$$

5.2.1 Binary Dilation (cont')

- dilation: union of translates of kernel

$$A \oplus B = \cup_{a \in A} B_a = \cup_{b \in B} A_b$$

- addition associative $\rightarrow$ dilation associative

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

- associativity of dilation: chain rule: iterative rule
- dilation of translated kernel: translation of dilation

$$A \oplus B_t = (A \oplus B)_t$$

5.2.1 Binary Dilation (cont')

- dilation distributes over union

$$(B \cup C) \oplus A = (B \oplus A) \cup (C \oplus A)$$

- dilating by union of two sets: the union of the dilation

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

5.2.1 Binary Dilation (cont')

- dilating *A* by kernel with origin guaranteed to contain *A*
- extensive: operators whose output contains input
- dilation extensive when kernel contains origin
- dilation preserves order

$$A \subseteq B \Longrightarrow A \oplus K \subseteq B \oplus K$$

- increasing: preserves order

5.2.2 Binary Erosion

- erosion: morphological dual of dilation
- erosion of *A* by *B:* set of all *x* s.t. $x + b \in A$ for every $b \in B$

$$A \ominus B = \{x \in E^N \mid x + b \in A \text{ for every } b \in B\}$$

- erosion: shrink: reduce

5.2.2 Binary Erosion (cont')

$$A = \{(1,0),\ (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\}$$

$$B = \{(0,0),\ (0,1)\}$$

$$A \ominus B = \{(1,0),\ (1,1),\ (1,2),\ (1,3),\ (1,4)\}$$

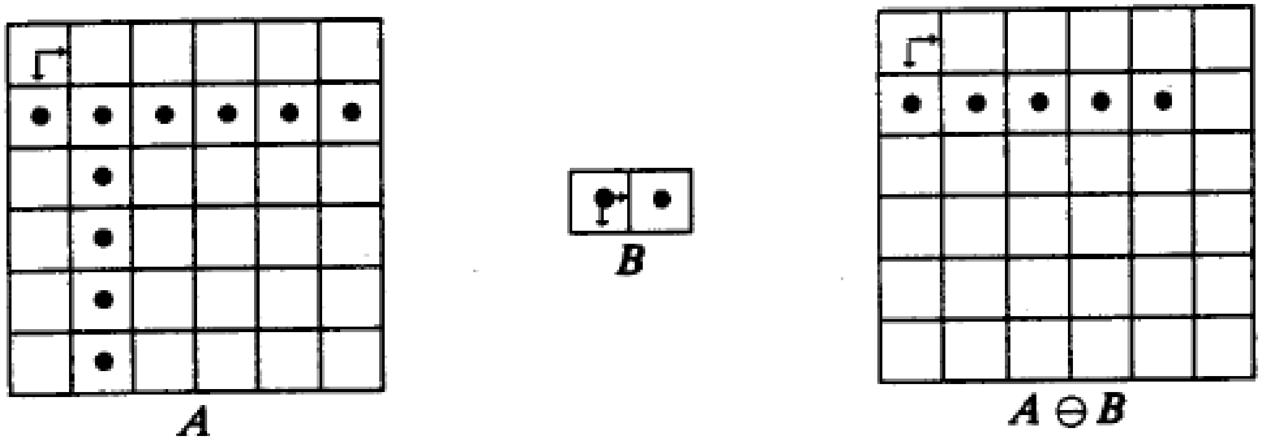
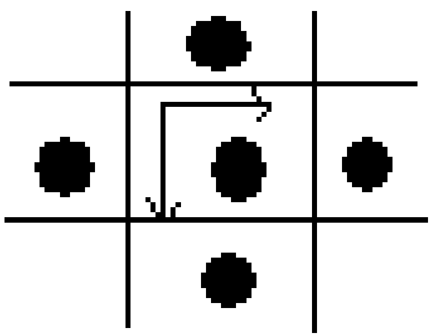


Figure 5.3 Erosion of a set, and its corresponding binary image.

5.2.2 Binary Erosion (cont')

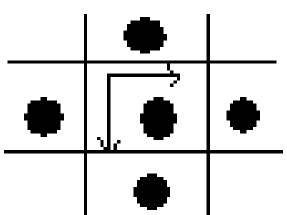
- =lena.bin.128=



5.2.2 Binary Erosion (cont')

- =Lena.bin.ero=

5.2.2 Binary Erosion (cont')



5.2.2 Binary Erosion (cont')

- erosion of *A* by *B*: set of all *x* for which *B* translated to *x* contained in *A*

$$A \ominus B = \{x \in E^N | B_x \subseteq A\}$$

- if *B* translated to *x* contained in *A* then *x* in $A \ominus B$
- erosion: difference of elements *a* and *b*

$$A \ominus B = \{x \in E^N | \text{ for every } b \in B, \text{ there exists an } a \in A \text{ such that } x = a - b\}$$

5.2.2 Binary Erosion (cont')

- dilation: union of translates
- erosion: intersection of negative translates

$$A \ominus B = \cap_{b \in B} A_{-b}$$

5.2.2 Binary Erosion (cont')

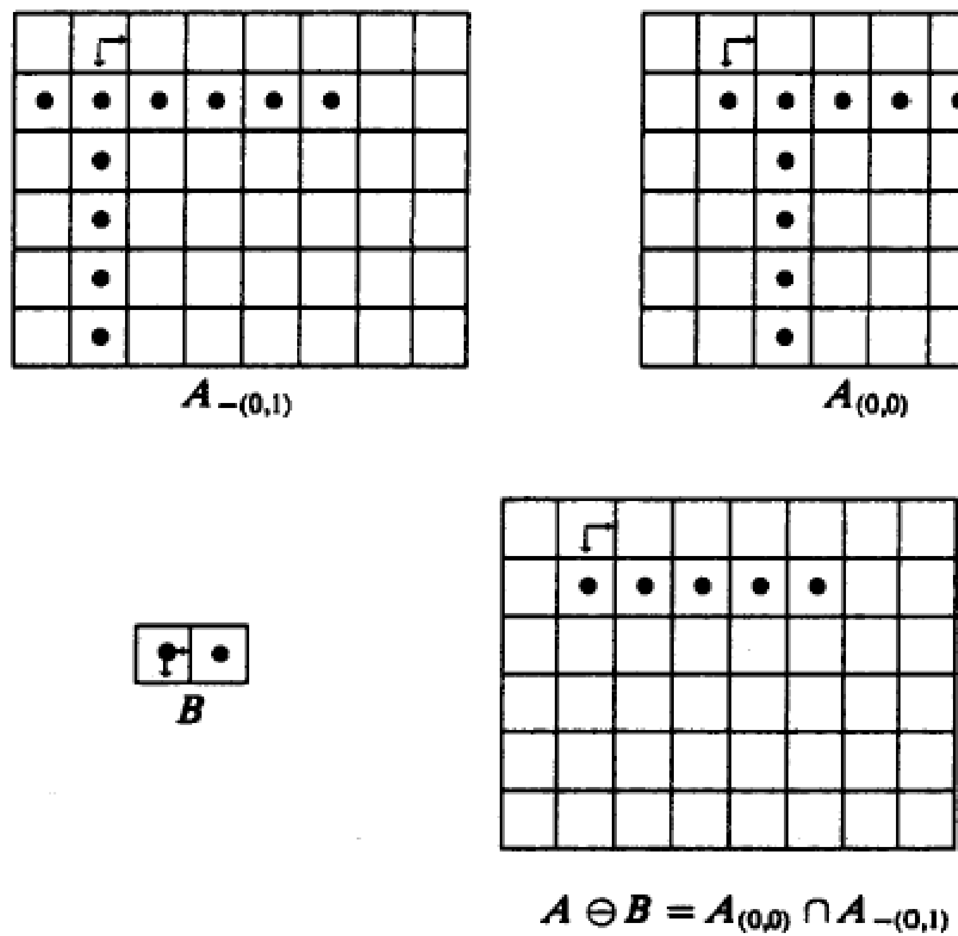


Figure 5.4 Erosion operation represented as the intersection of translated sets.

5.2.2 Binary Erosion (cont')

- Minkowski subtraction: close relative to erosion
- Minkowski subtraction: $\bigcap_{b \in B} A_b$
- erosion: shrinking of the original image
- antiextensive: operated set contained in the original set

5.2.2 Binary Erosion (cont')

- if $0 \in B$ then $A \ominus B \subseteq A$ because

if $x \in A \ominus B$ then $x + b \in A$ for every $b \in B$, since $0 \in B$ thus $x = x + 0 \in A$

- eroding *A* by kernel without origin can have nothing in common with *A*

5.2.2 Binary Erosion (cont')

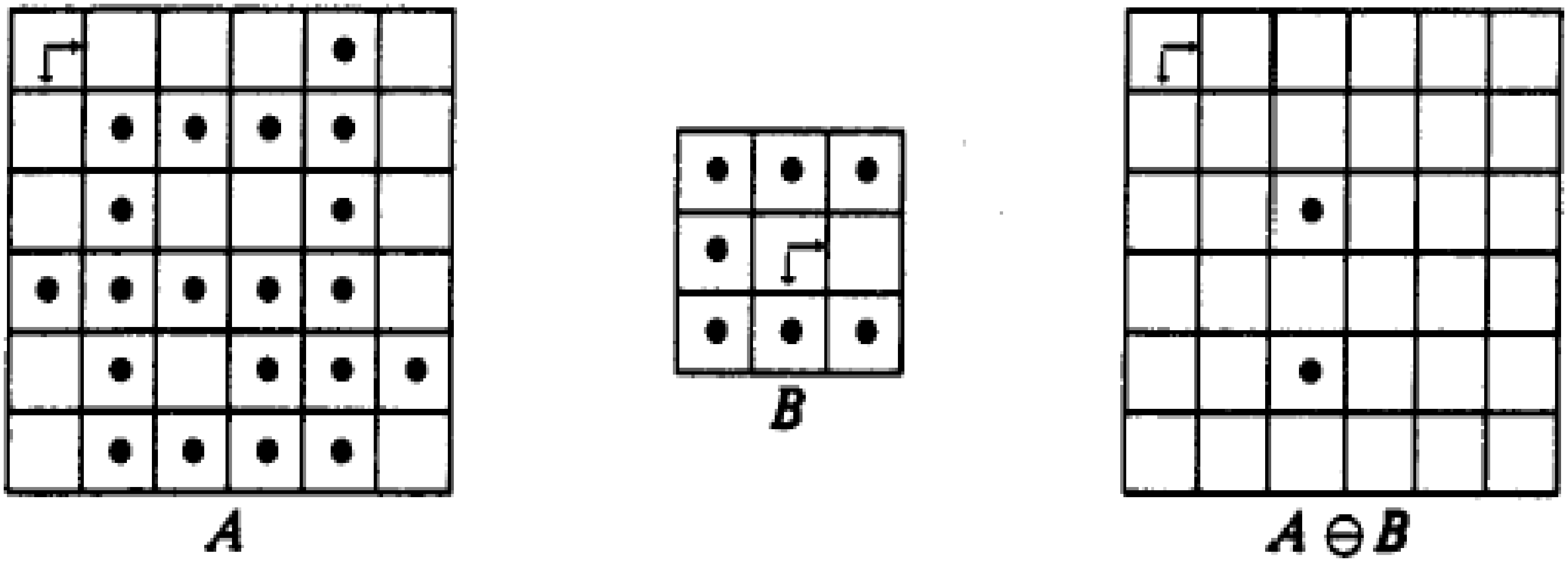


Figure 5.5 Erosion of a set A by a structuring element B that does not contain the origin. As a result, no point of the erosion is guaranteed to be in common with A. However, some translations of $A \ominus B$ are contained in A.

5.2.2 Binary Erosion (cont')

possible: $A \ominus B \subseteq A$ and $0 \notin B$

e.g. $A = \{1,2,3,4\}, B = \{-1,1\}$, then $A \ominus B = \{2,3\} \subseteq A$, yet $0 \notin B$

- dilating translated set results in a translated dilation

$$A_t \oplus B = (A \oplus B)_t$$

- eroding by translated kernel results in negatively translated erosion $A \ominus B_t = (A \ominus B)_{-t}$

- dilation, erosion: increasing $A \subseteq B \Longrightarrow A \ominus K \subseteq B \ominus K$

5.2.2 Binary Erosion (cont')

- eroding by larger kernel produces smaller result

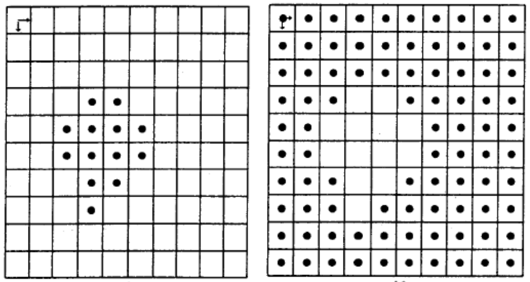
$$K \subseteq L \Longrightarrow A \ominus L \subseteq A \ominus K$$

- Dilation, erosion similar that one does to foreground, the other to background
- similarity: duality
- dual: negation of one equals to the other on negated variables
- DeMorgan's law: duality between set union and intersection $(A \cup B)^c = A^c \cap B^c$

5.2.2 Binary Erosion (cont')

- negation of a set: complement

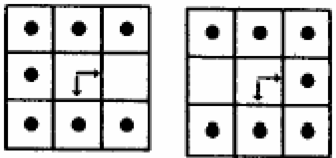
$$A^c = \{x \in E^n | x \notin A\}$$



A A^c

- negation of a set in two possible ways in morphology
  - logical sense: set complement
  - geometric sense: reflection: reversing of set orientation

$\check{B}$: reflection about the origin of $B \subseteq E^N$



B $\check{B}$

$$\check{B} = \{x | \text{for some } b \in B, x = -b\}$$

5.2.2 Binary Erosion (cont')

$\check{B}$: symmetrical set of B with respect to origin [Matheron 1975]
$\check{B}$: B transpose [Serra 1982]

- complement of erosion: dilation of the complement by reflection
- Theorem 5.1: Erosion Dilation Duality

$$(A \ominus B)^c = A^c \oplus \check{B}$$

5.2.2 Binary Erosion (cont')

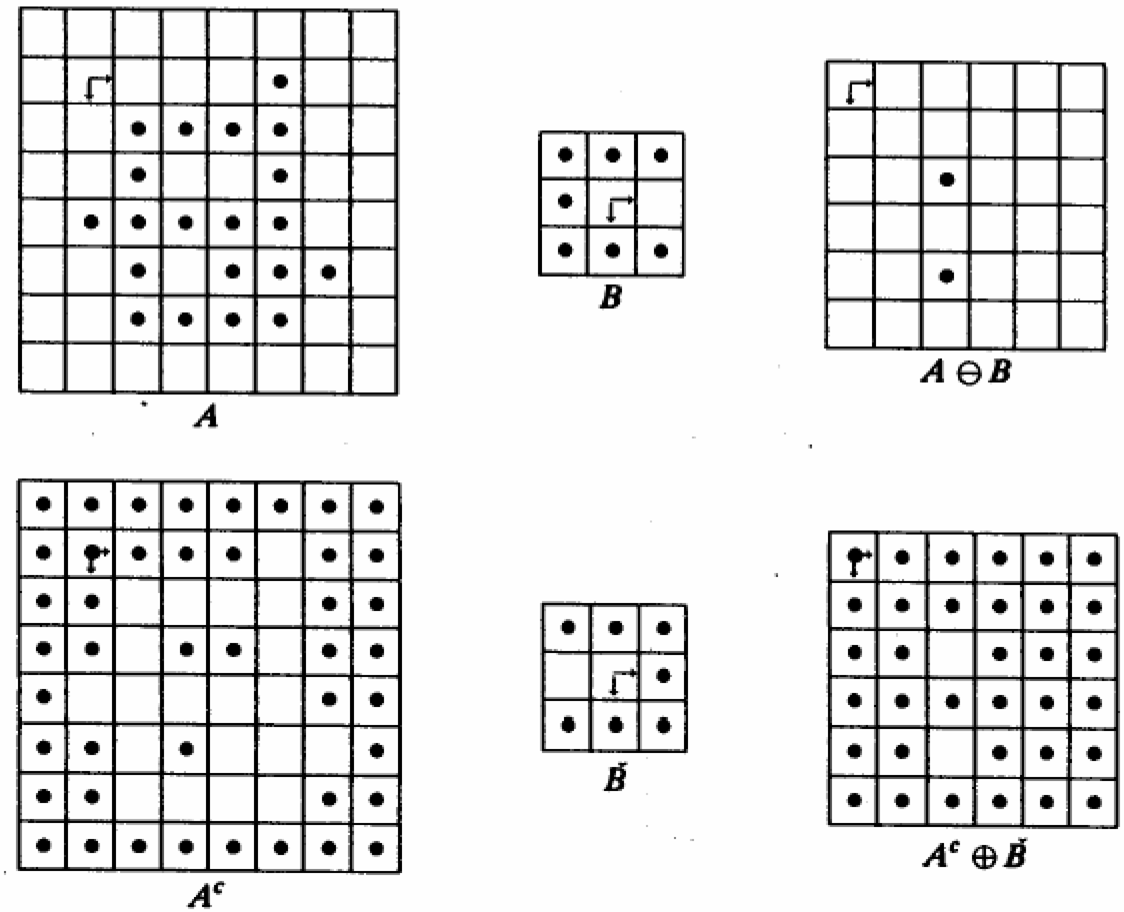


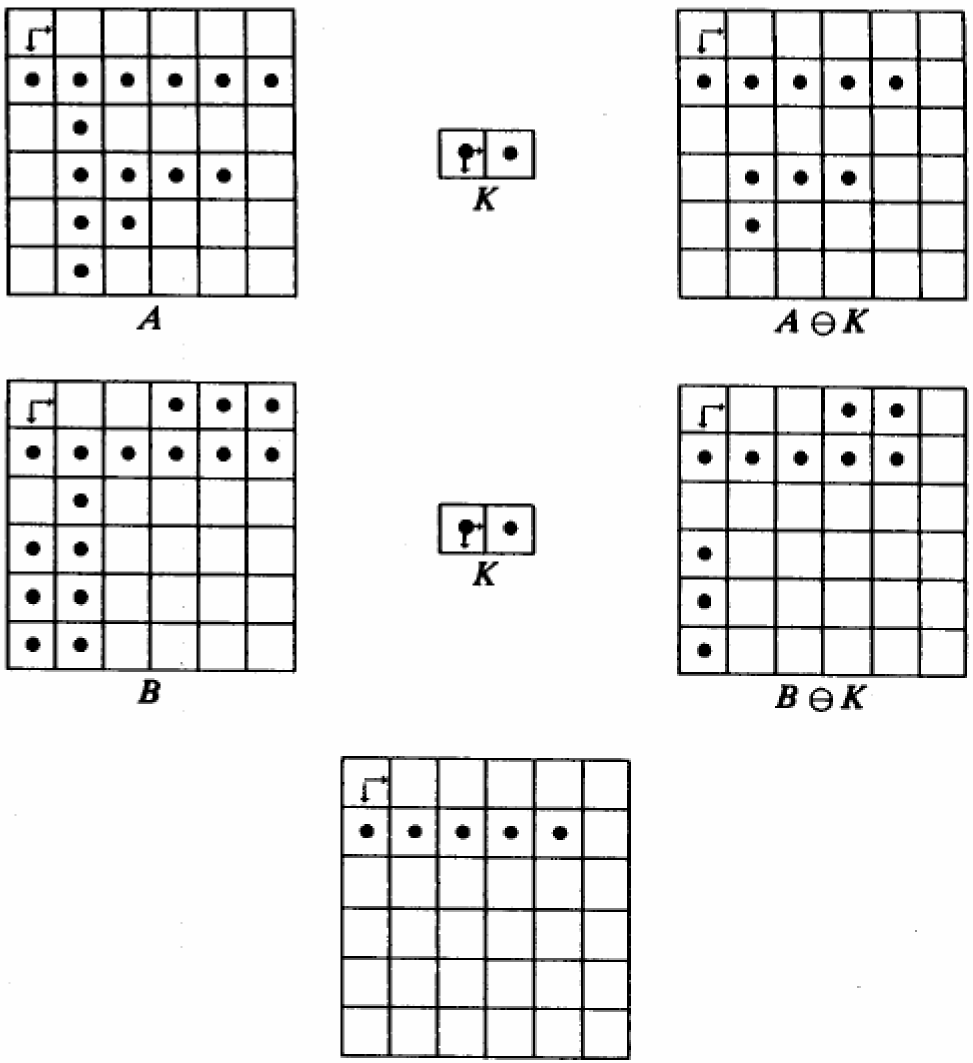
Figure 5.6 Duality relation between erosion and dilation. The set A eroded by B is the complement of the set A^c dilated by $\check{B}$. By convention, we understand that for the complemented set A^c or $A^c \oplus \check{B}$, all pixels outside the area illustrated are binary-1 pixels.

5.2.2 Binary Erosion (cont')

- Corollary 5.1: $(A \oplus B)^c = A^c \ominus \check{B}$
- erosion of intersection of two sets: intersection of erosions

$$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$$

5.2.2 Binary Erosion (cont')



$$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$$

Figure 5.7 An instance of the relationship $(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$.

5.2.2 Binary Erosion (cont')

- erosion of a kernel of union of two sets: intersection of erosions

$$A \ominus (K \cup L) = (A \ominus K) \cap (A \ominus L)$$

- erosion of kernel of intersection of two sets: contains union of erosions

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C)$$

- no stronger

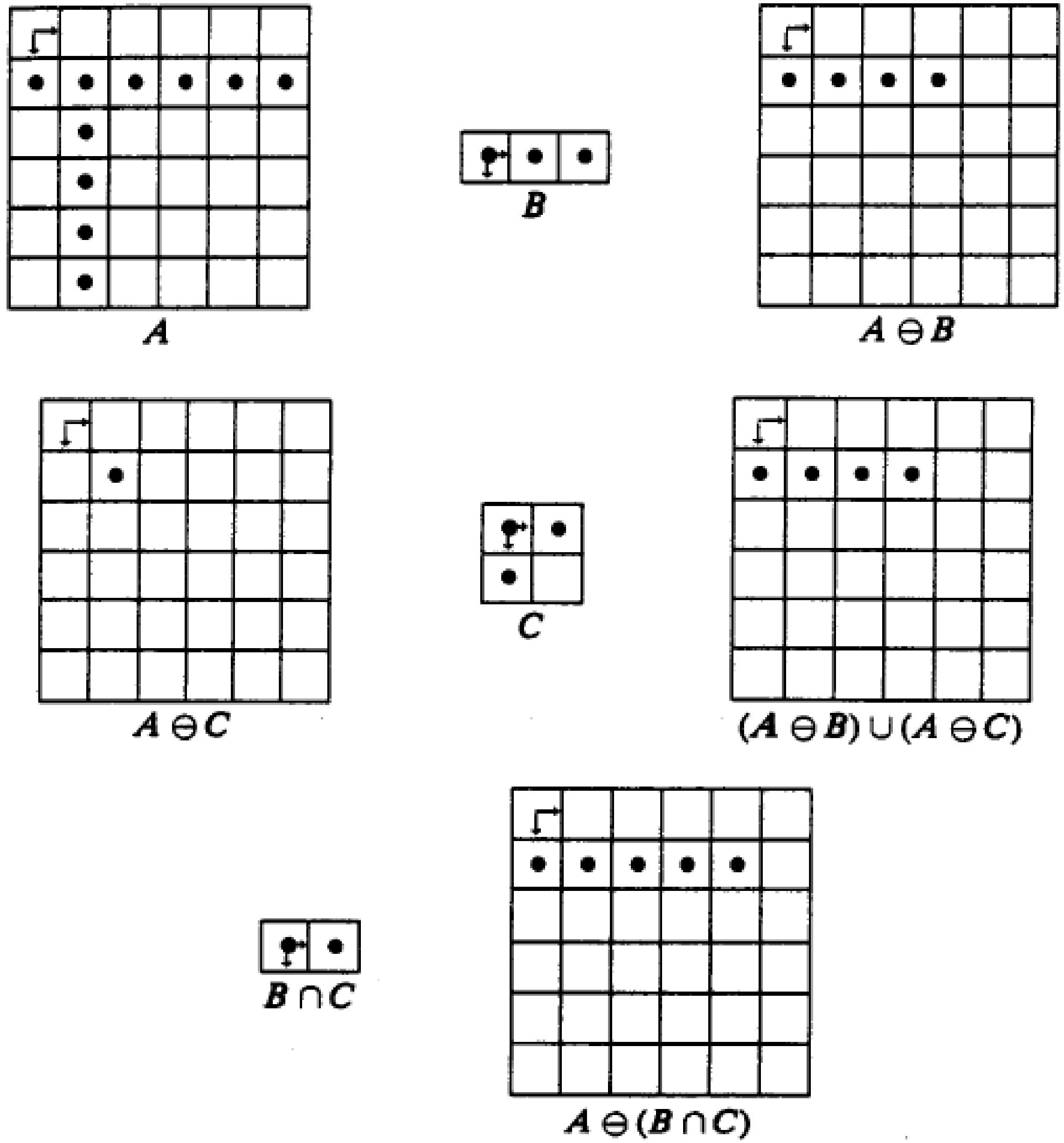


Figure 5.8 An instance in which $A \ominus (B \cap C)$ is a proper superset of $(A \ominus B) \cup (A \ominus C)$, thereby showing that the general relation $A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C)$ cannot be made any stronger.

5.2.2 Binary Erosion (cont')

- chain rule for erosion holds when kernel decomposable through dilation

$$A \ominus (B \oplus C) = (A \ominus B) \ominus C$$

- duality does not imply cancellation on morphological equalities

$$(B \ominus C) \oplus C \neq B$$

- containment relationship holds

$$A \subseteq B \ominus C \text{ if and only if } B \supseteq A \oplus C$$

5.2.2 Binary Erosion (cont')

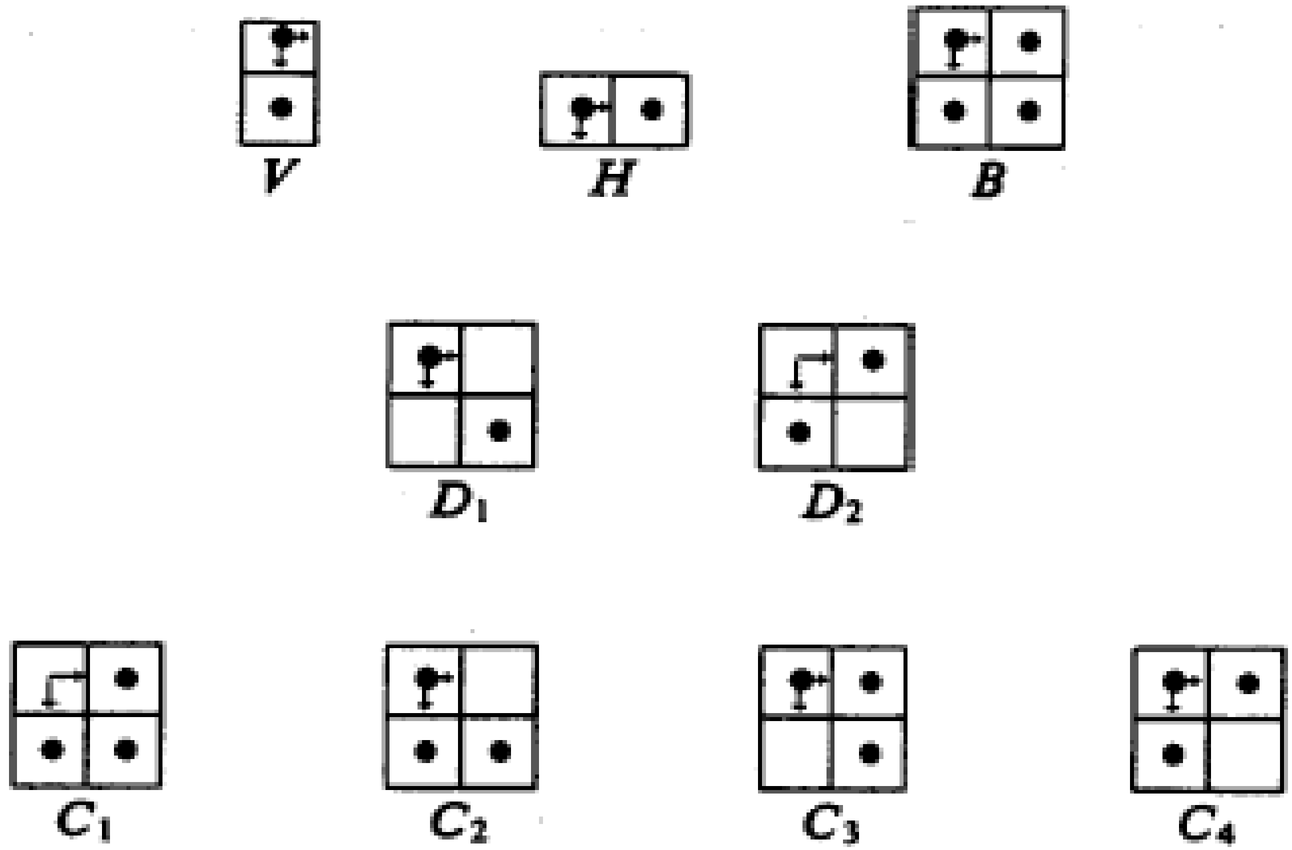
- genus *g(I):* number of connected components minus number of holes of *I,* 4-connected for object, 8-connected for background

$$g_4(I) = \#I - \#I \ominus V - \#I \ominus H + \#I \ominus B$$

- 8-connected for object, 4-connected for background

$$g_8(I) = \#I - \#I \ominus V - \#I \ominus H - \#I \ominus D_1 - \#I \ominus D_2 + \#I \ominus C_1 + \#I \ominus C_2 + \#I \ominus C_3$$
$$+ \#I \ominus C_4 - \#I \ominus B$$

5.2.2 Binary Erosion (cont')



5.2.2 Binary Erosion (cont')

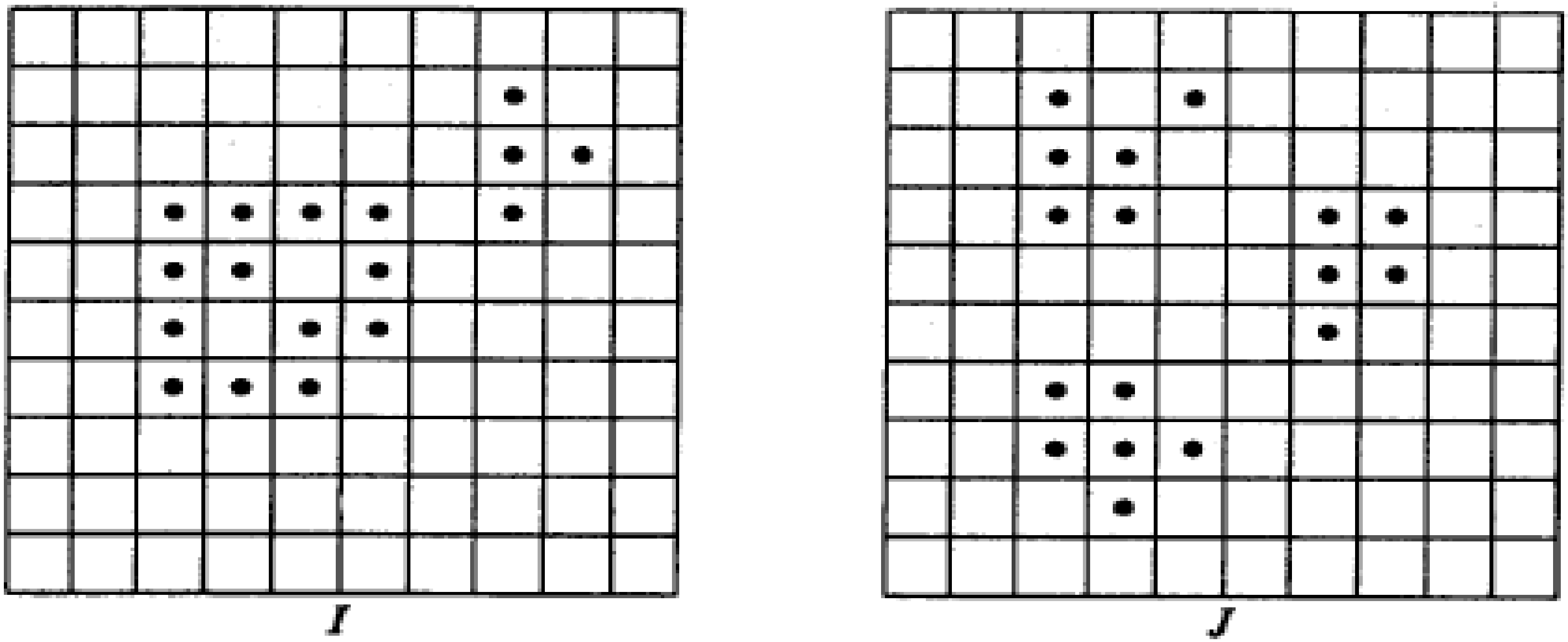


Figure 5.9 Structuring elements that can be used to determine the genus of a binary image, and two examples with this genus computed according to Eqs. (5.1) and (5.2): $g_4(I) = 1$, $g_4(J) = 4$, $g_8(I) = 0$, $g_8(J) = 3$.

5.2.3 Hit-and-Miss Transform

- hit-and-miss: selects corner points, isolated points, border points
- hit-and-miss: performs template matching, thinning, thickening, centering
- hit-and-miss: intersection of erosions
- *J*,*K* kernels satisfy $J \cap K = \emptyset$
- hit-and-miss of set *A* by (*J*,*K*)

$$A \otimes (J, K) = (A \ominus J) \cap (A^c \ominus K)$$

- hit-and-miss: to find upper right-hand corner

5.2.3 Hit-and-Miss Transform (cont')

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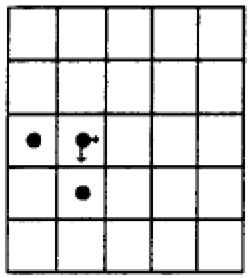
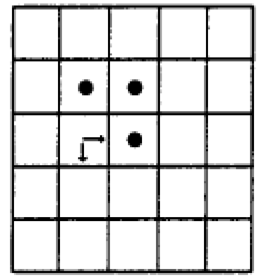
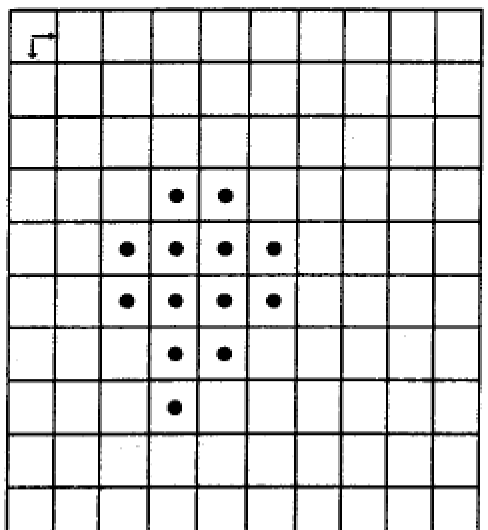


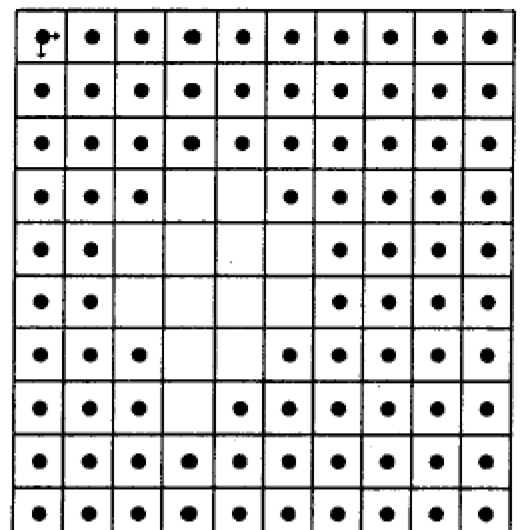
J



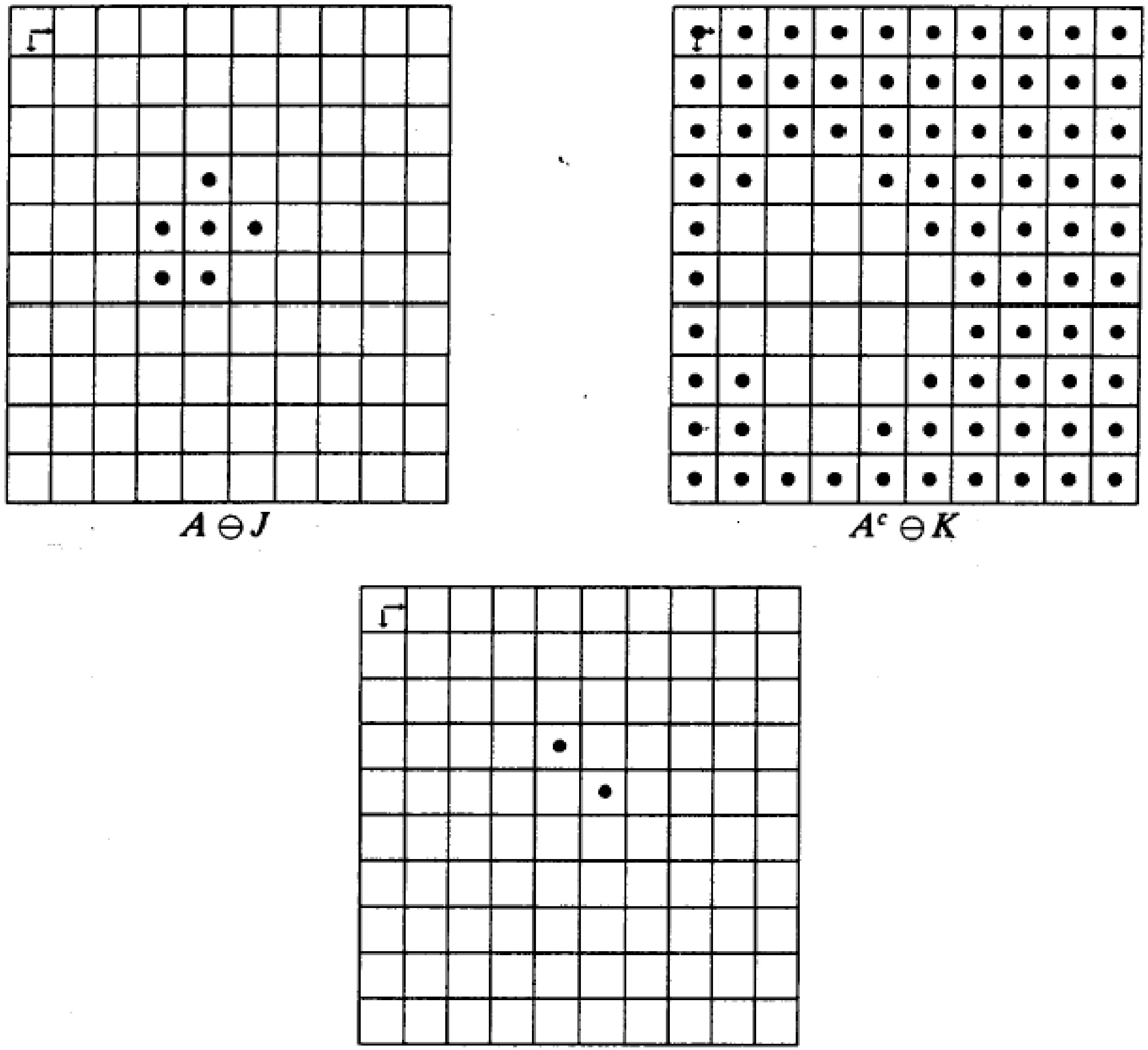
K



A



A^c



$$A \otimes (J,K) = (A \ominus J) \cap (A^c \ominus K)$$

Figure 5.11 Location of upper right-hand corner points by the hit-and-miss transform.

5.2.3 Hit-and-Miss Transform (cont')

- J locates all pixels with south, west neighbors of A
- K locates all pixels of A^c with south, west neighbors in A^c
- J and K displaced from one another
- Hit-and-miss: locate particular spatial patterns $J = \{(0,0)\}, K = \{(0,1),(0,-1),(1,0),(-1,0)\}$

$$\text{then } I \otimes (J, K)\text{: set of all 4-isolated pixels}$$

5.2.3 Hit-and-Miss Transform (cont')

- hit-and-miss: to compute genus of a binary image

$$g_4(I) = \#I \otimes (J_1, K_1) + \#I \otimes (J_2, K_2) - \#I \otimes (J_3, K_3)$$

$$g_8(I) = \#I \otimes (J_1, K_1) - \#I \otimes (J_4, K_4)$$

5.2.3 Hit-and-Miss Transform (cont')

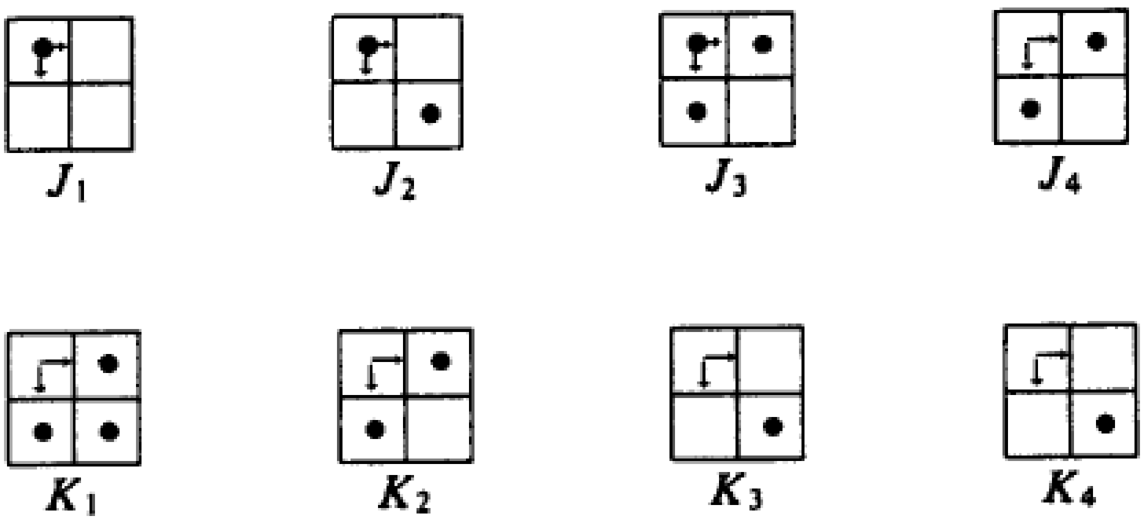


Figure 5.10 Structuring element by which genus can be computed via a hit-and-miss transform.

5.2.3 Hit-and-Miss Transform (cont')

- hit-and-miss: thickening and thinning
- hit-and-miss: counting
- hit-and-miss: template matching

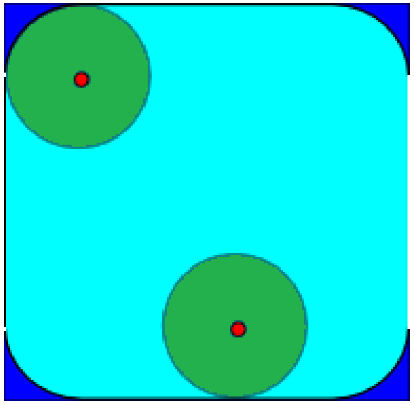
5.2.5 Opening and Closing

- dilation and erosion: usually employed in pairs
- *B*○*K*: opening of image *B* by kernel *K*

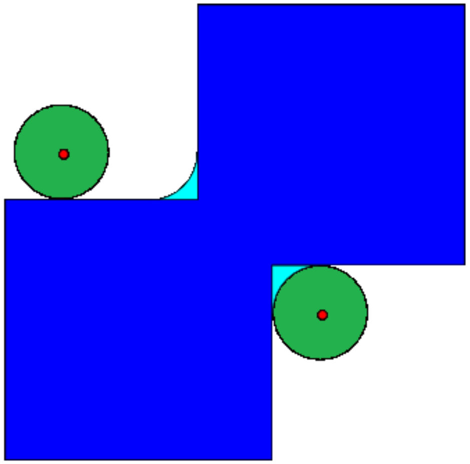
$$B \circ K = (B \ominus K) \oplus K$$

- *B*•*K:* closing of image *B* by kernel *K*

$$B \bullet K = (B \oplus K) \ominus K$$



Opening



closing

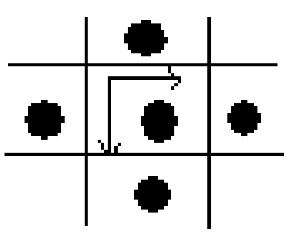
5.2.5 Opening and Closing (cont')

- =lena.bin.128=

5.2.5 Opening and Closing (cont')

- =lena.bin.open=

5.2.5 Opening and Closing (cont’)



Opening

5.2.5 Opening and Closing (cont')

- morphological opening, closing: no relation to topologically open, closed sets
- opening characterization theorem

$$A \circ K = \{x \in A \mid \text{for some } t \in A \ominus K, x \in K_t \text{ and } K_t \subseteq A\}$$

- $A \circ K$: selects points covered by some translation of K, entirely contained in A

5.2.5 Opening and Closing (cont')

- opening with disk kernel: smoothes contours, breaks narrow isthmuses
- opening with disk kernel: eliminates small islands, sharp peaks, capes
- closing by disk kernel: smoothes contours, fuses narrow breaks, long, thin gulfs
- closing with disk kernel: eliminates small holes, fill gaps on the contours

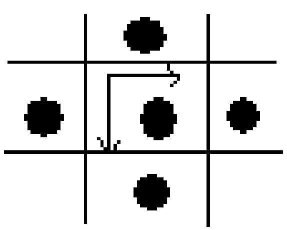
5.2.5 Opening and Closing (cont')

- =lena.bin.128=

5.2.5 Opening and Closing (cont')

- =lena.bin.close=

5.2.5 Opening and Closing (cont')



5.2.5 Opening and Closing (cont')

- unlike erosion and dilation: opening invariant to kernel translation

$$A \circ K = A \circ K_x$$

- opening antiextensive $A \circ K \subseteq A$
- like erosion and dilation: opening increasing

$$A \subseteq B \Longrightarrow A \circ K \subseteq B \circ K$$

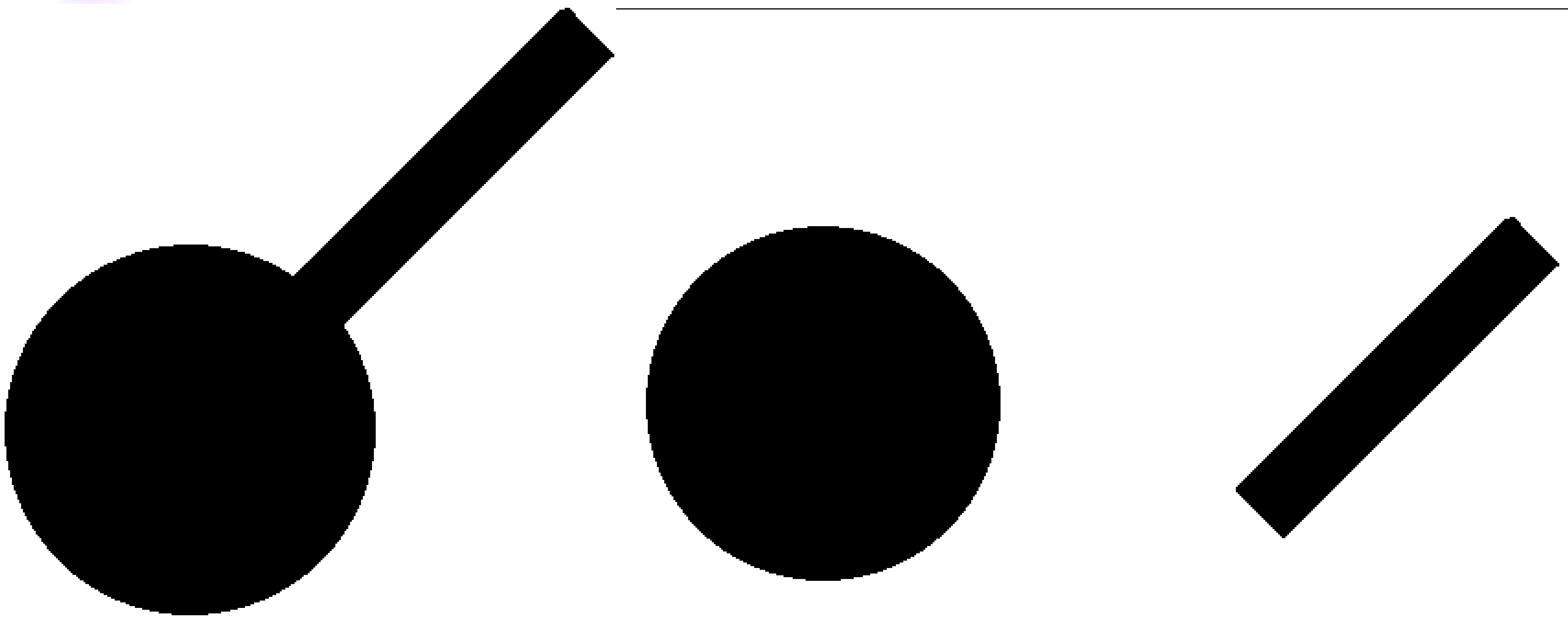
5.2.5 Opening and Closing (cont')

- $A \circ K$: those pixels covered by sweeping kernel all over inside of A

$$A \circ K = \bigcup_{y \in A \ominus K} K_y = \bigcup_{K_y \subseteq A} K_y$$

- *F:* shape with body and handle
- *L*: small disk structuring element with radius just larger than handle width
- extraction of the body and handle by opening and taking the residue

5.2.5 Opening and Closing (cont')



(a) (b) (c)

Figure 5.16 Extraction of the body and handle of a shape F by opening with L for the body and taking the residue of the opening for the handle; (a) F, (b) $F \circ L$, and (c) $F - F \circ L$.

5.2.5 Opening and Closing (cont')

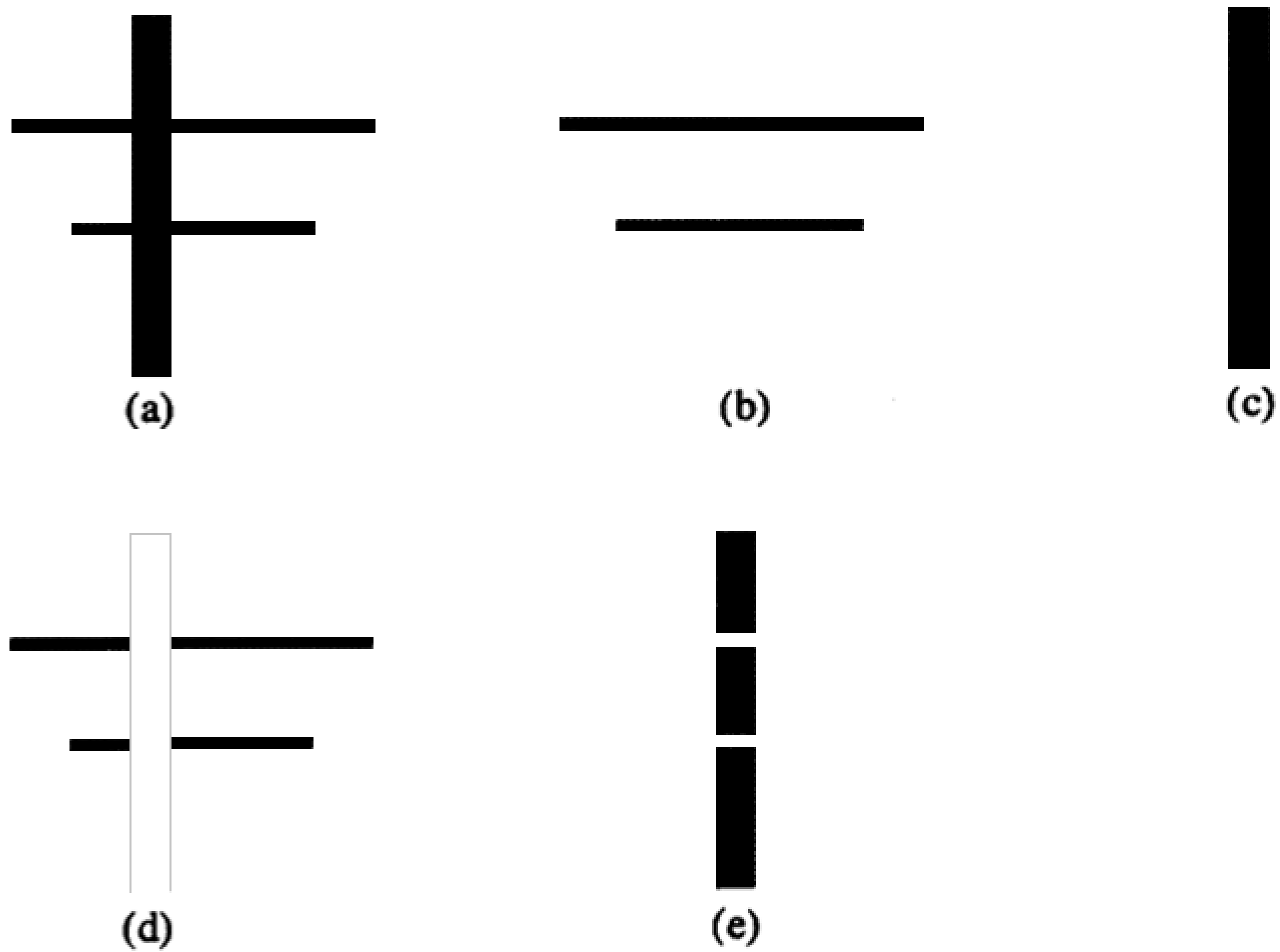
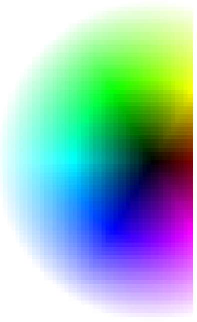
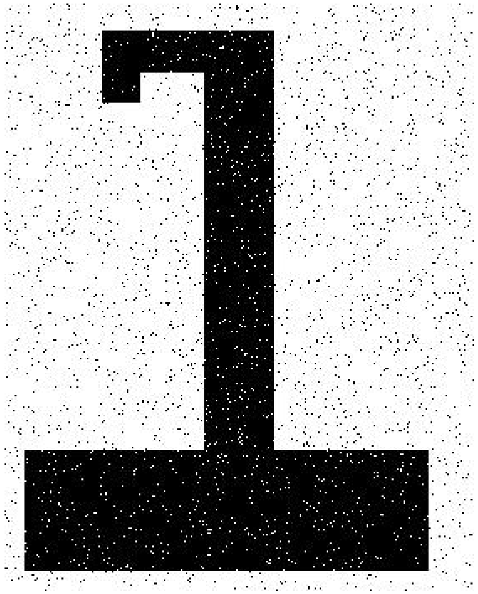


Figure 5.17 Extraction of the trunk and arms of a shape *F* by opening with vertically and horizontally oriented structuring elements.

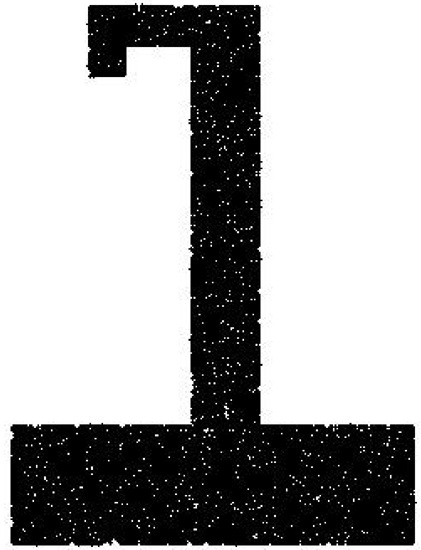
5.2.5 Opening and Closing (cont')

- extraction of trunk and arms with vertical and horizontal kernels

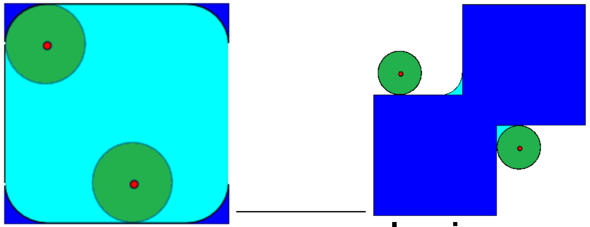




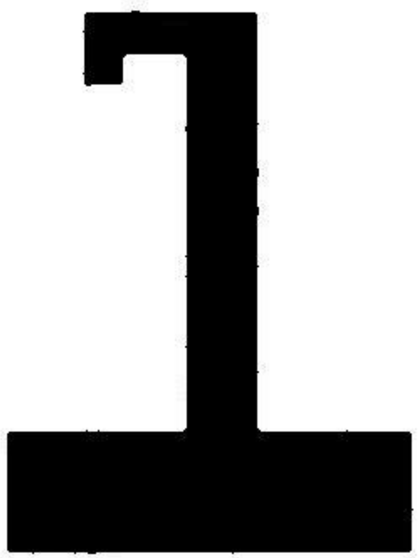
(a)



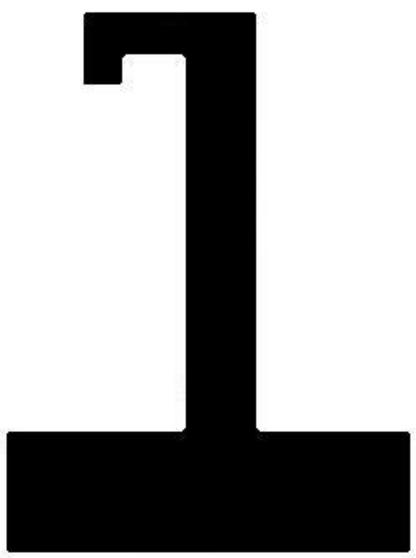
(b)



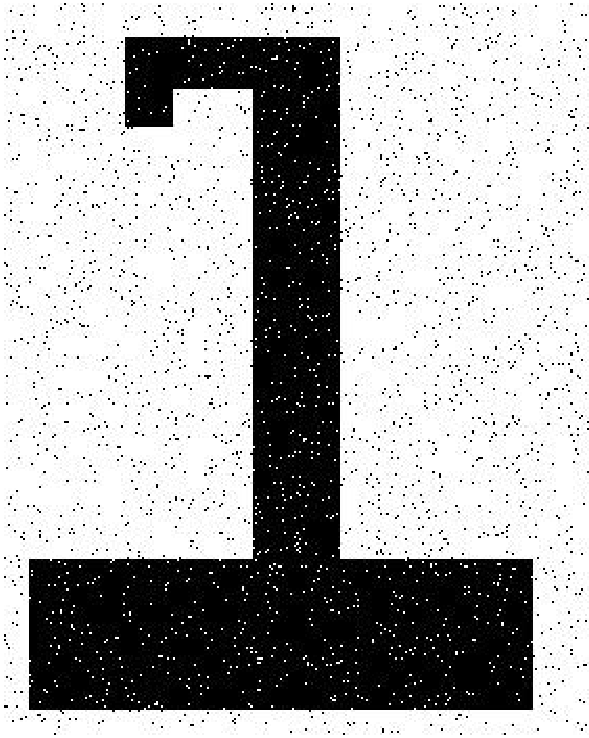
Opening closing



(c)



(d)



(a)~(d)

5.2.5 Opening and Closing (cont')

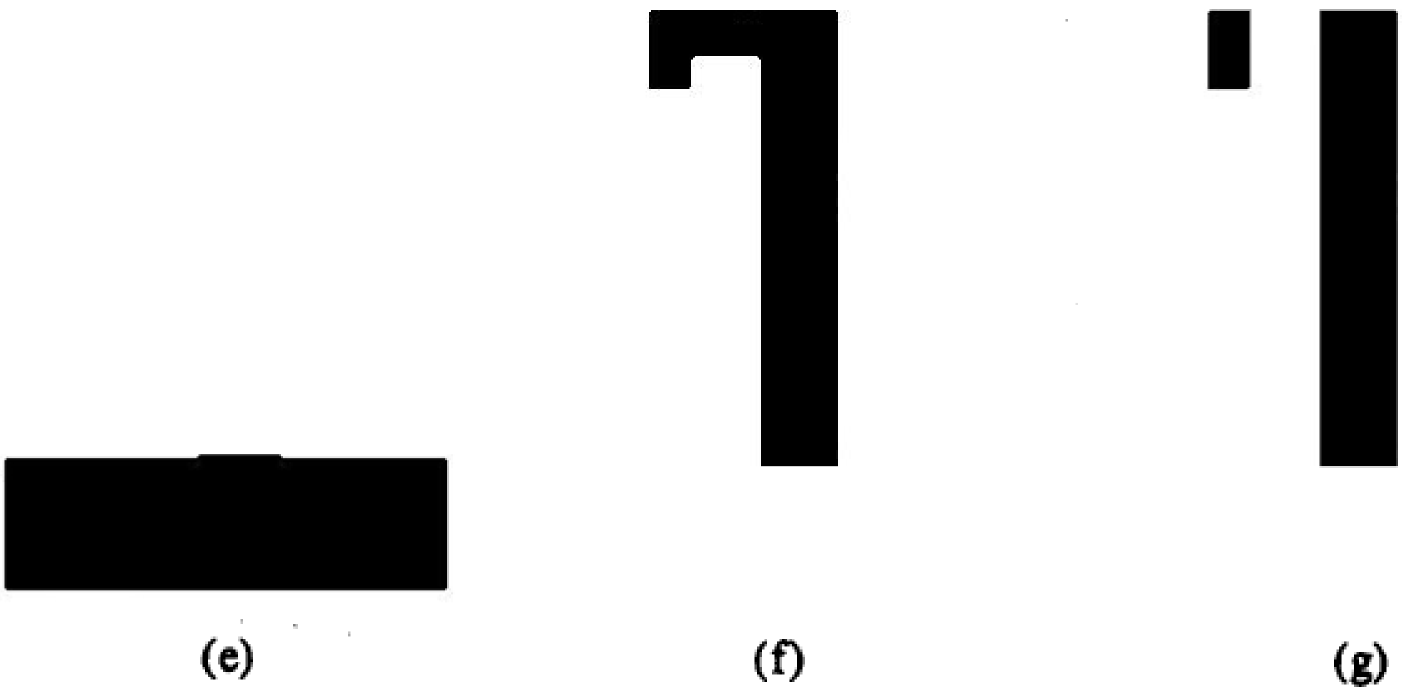
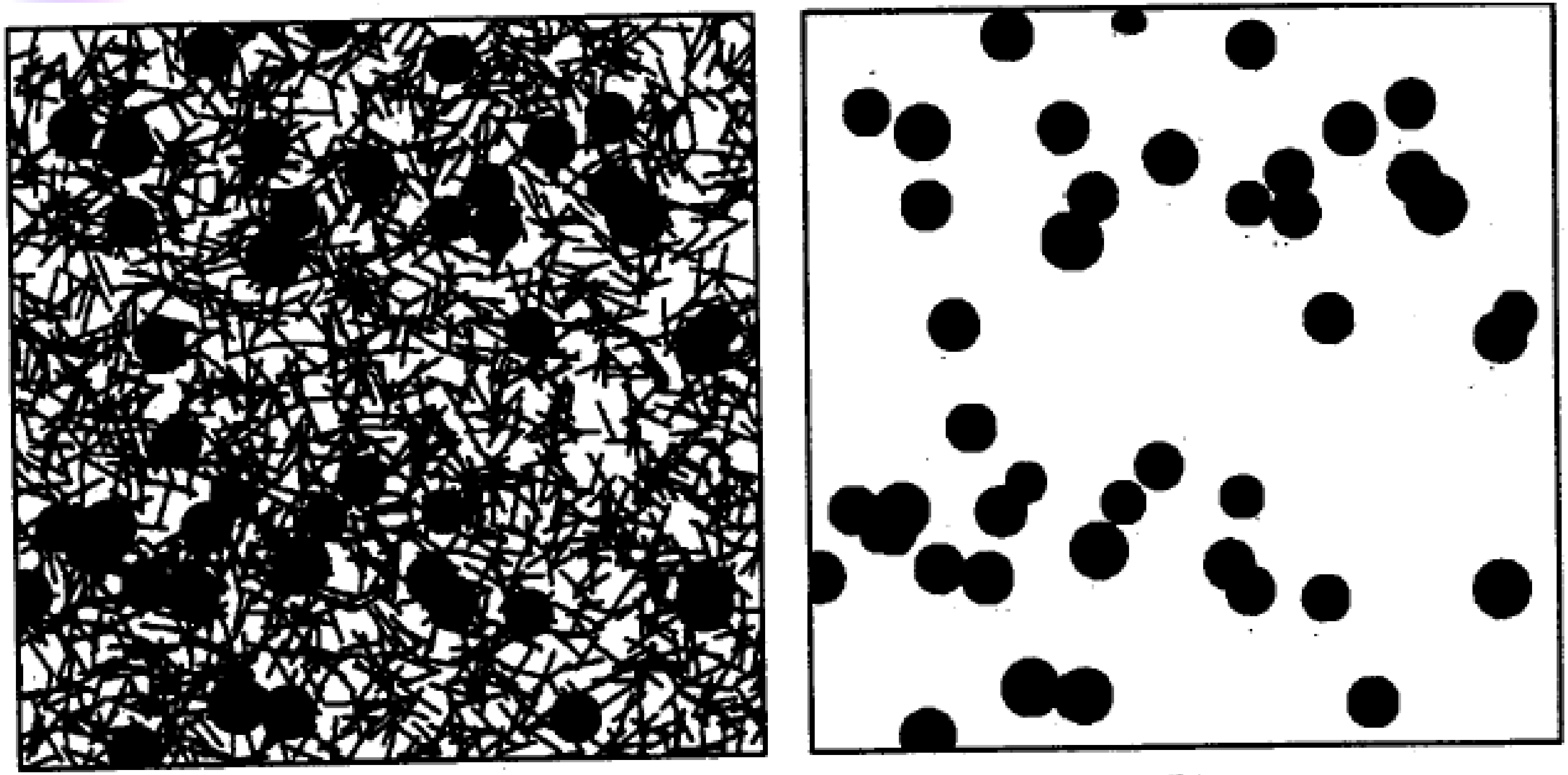


Figure 5.18 Extraction of the base, trunk, and horizontal and vertical areas of a shape *F* immersed in salt-and-pepper background noise by conditioning and then opening; (a) original image, (b) opening with a disk of radius 1, (c) closing with a disk of radius 4, (d) opening with a disk of radius 3, (e) opening with a rectangle of size 21×20, (f) residue of the opening, and (g) opening residue of opening with a vertical structuring element.

5.2.5 Opening and Closing (cont’)

- extraction of base, trunk, horizontal and vertical areas

5.2.5 Opening and Closing (cont')

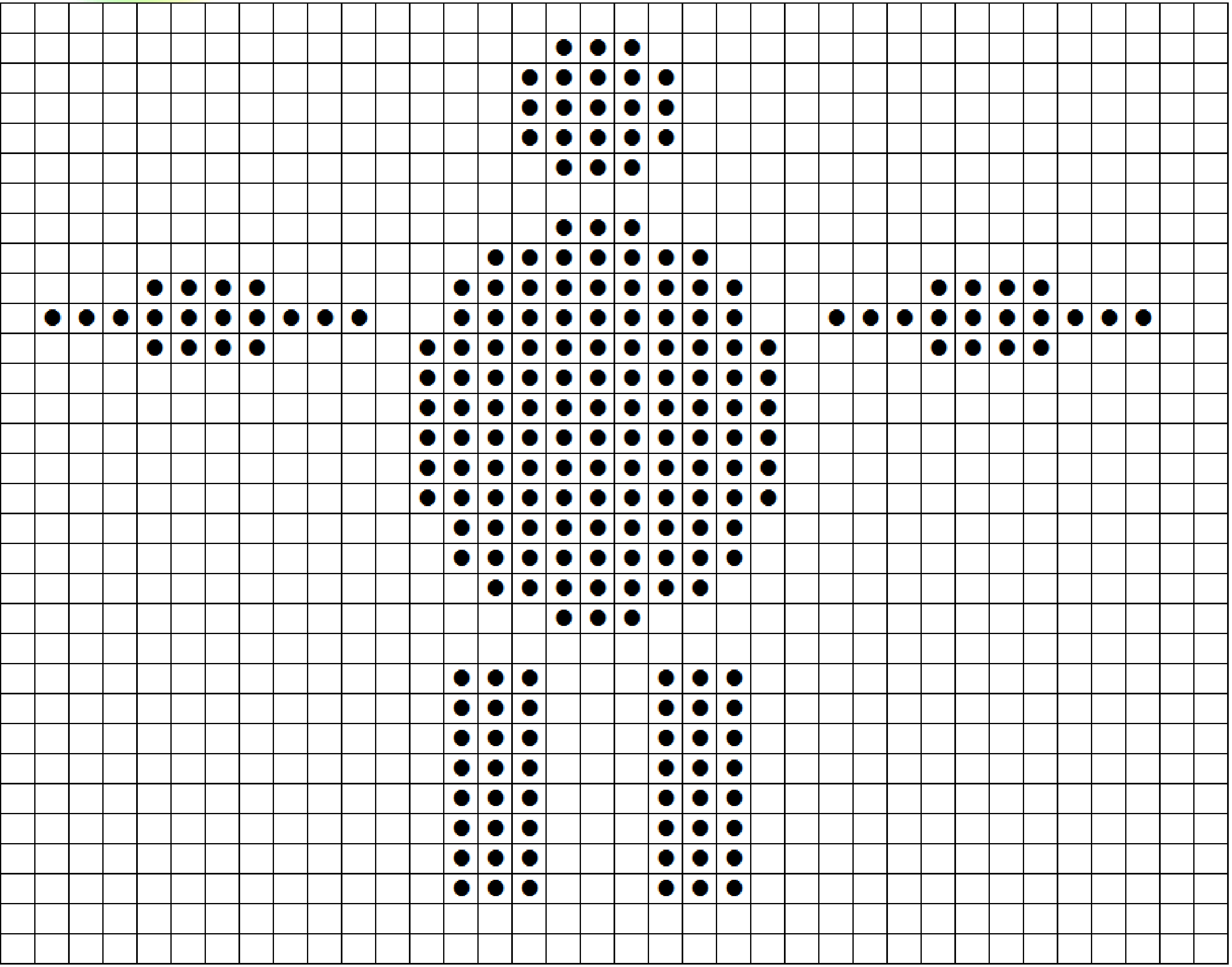


(a) (b)

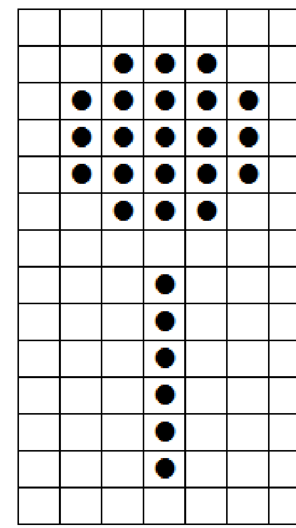
Figure 5.19 (a) A binary image. (b) Opening of (a) with a disk structuring element.

5.2.5 Opening and Closing (cont')

- noisy background line segment removal

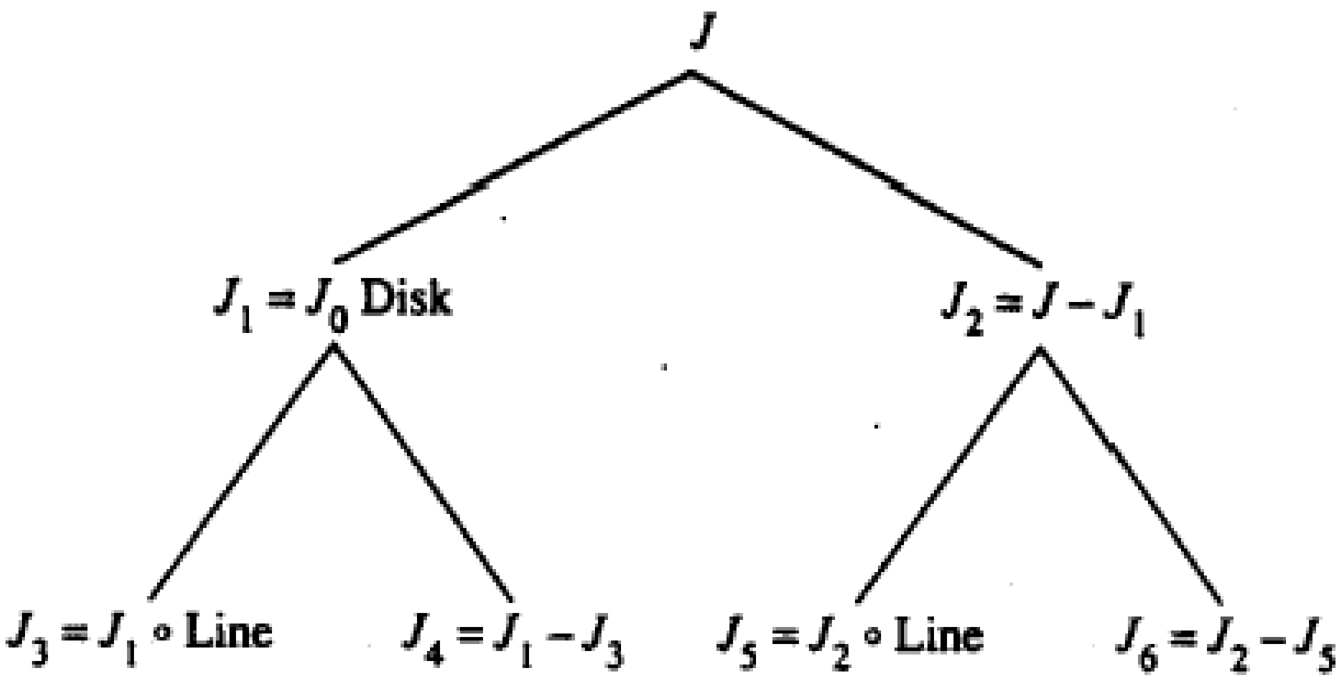


(a) The image *J*.



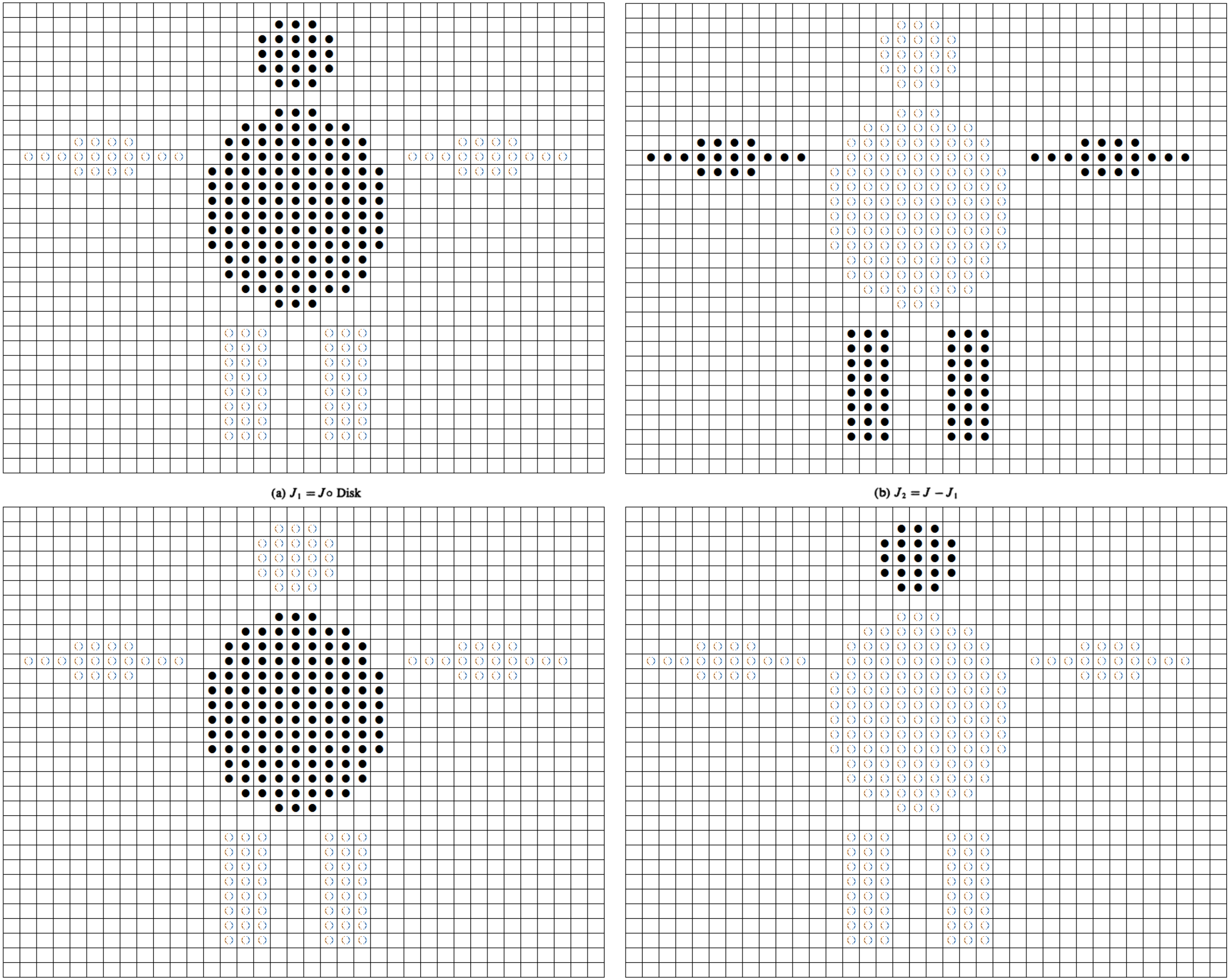
(b)

5.2.5 Opening and Closing (cont')



(c)

Figure 5.20 (a) An image. (b) A disk and line structuring element. (c) The opening-sequence tree decomposition operations for an image.



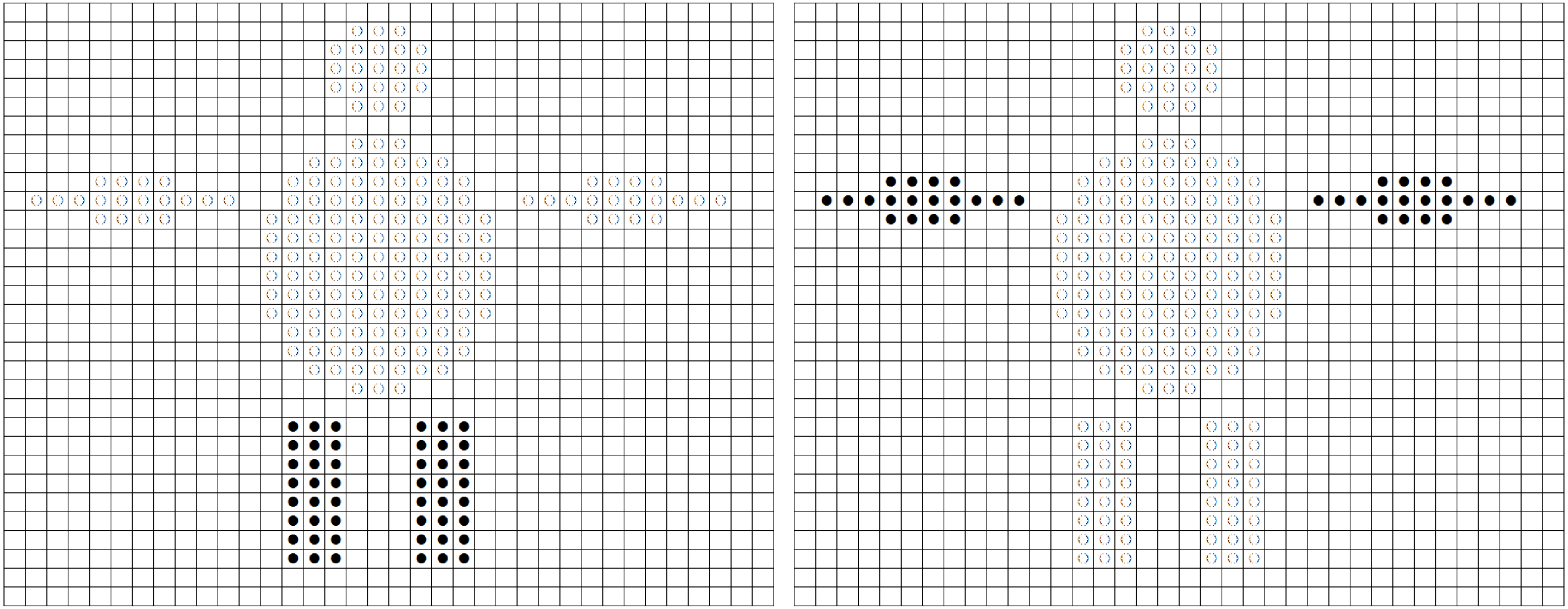
(a) $J_1 = J \circ \text{Disk}$

(b) $J_2 = J - J_1$

(c) $J_3 = J_1 \circ \text{Line}$

(d) $J_4 = J_3 - J_1$

5.2.5 Opening and Closing (cont')



(e) $J_5 = J_2 \circ$ Line

(f) $J_6 = J_2 - J_5$

Figure 5.21 Results of the opening-sequence tree decomposition operations applied to image J of Fig. 5.20(a). The $\circ$ designates a point in J that is not in the illustrated set. A $\bullet$ designates a point in the illustrated set.

5.2.5 Opening and Closing (cont')

- decomposition into parts

5.2.5 Opening and Closing (cont')

- closing: dual of opening

$$A \bullet K = \{x | x \in \check{K}_t \text{ implies } \check{K}_t \cap A \neq \emptyset\}$$

- like opening: closing invariant to kernel translation $A \bullet K_x = A \bullet K$
- closing extensive $A \subseteq A \bullet K$
- like dilation, erosion, opening: closing increasing

$$A \subseteq B \Longrightarrow A \bullet K \subseteq B \bullet K$$

5.2.5 Opening and Closing (cont')

- opening idempotent

$$(A \circ K) \circ K = A \circ K$$

- closing idempotent

$$(A \bullet K) \bullet K = A \bullet K$$

- if $L \subseteq K$ not necessarily follows that $A \circ L \supseteq A \circ K$

5.2.5 Opening and Closing (cont')

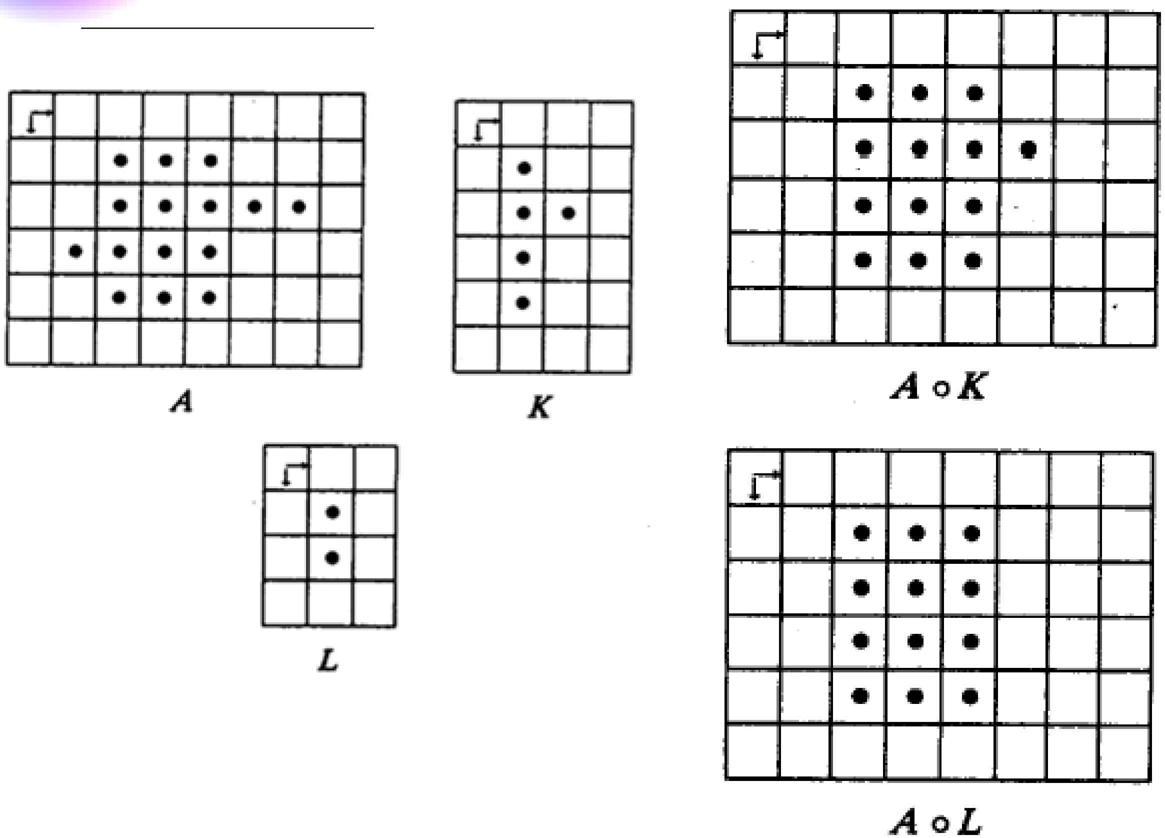


Figure 5.22 Two structuring elements K and L, where $L \subseteq K$. The set A is opened by K and by L. In this instance the opening by the large structuring element produces the larger result.

5.2.5 Opening and Closing (cont')

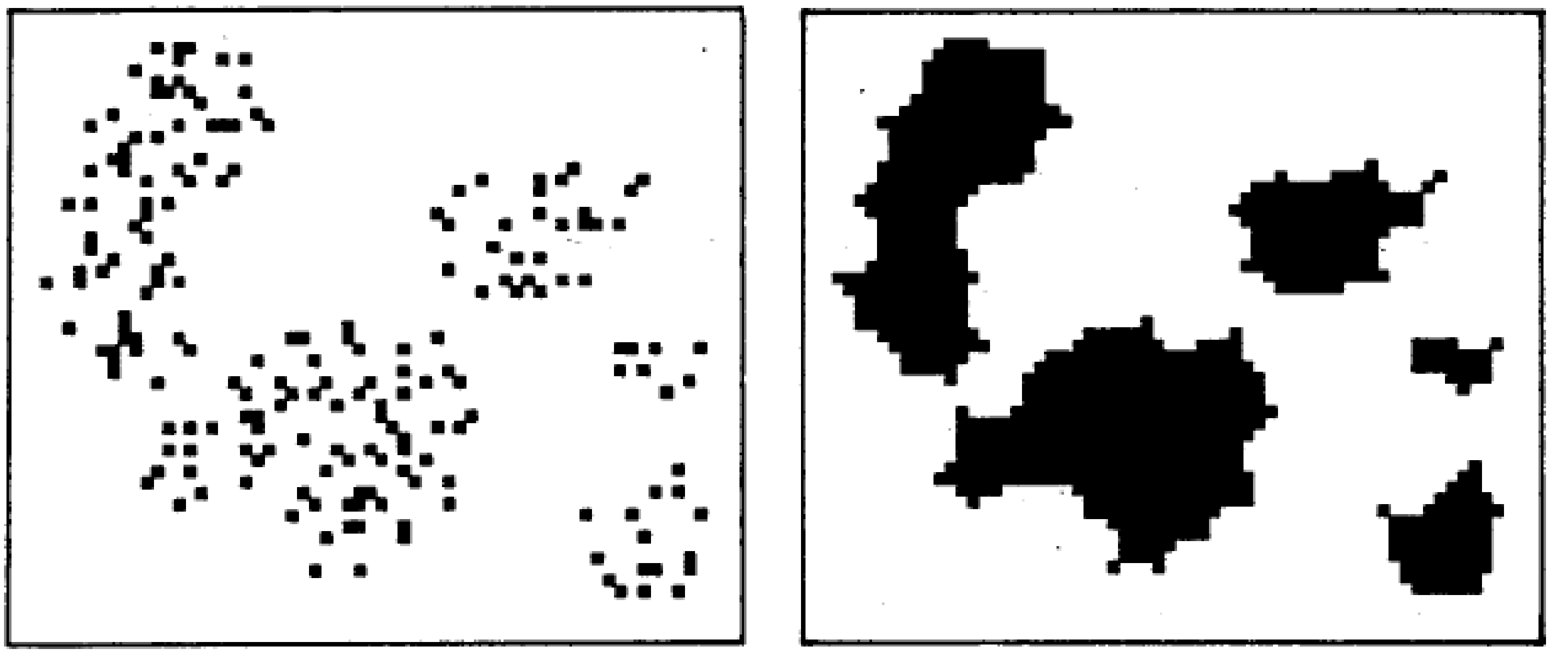

(a) (b)

Figure 5.23 (a) A binary image with five clusters of points. Points within each cluster satisfy the partition property with distance ρ_0, and the clusters are farther from each other than $2\rho_0$ pixels. (b) The image of (a) closed by a disk with a radius just greater than $2\rho_0$.

5.2.5 Opening and Closing (cont')

- closing may be used to detect spatial clusters of points

5.3 Connectivity

- morphology and connectivity: close relation

5.3.1 Separation Relation

- *S* separation if and only if *S* symmetric, exclusive, hereditary, extensive

5.3.2 Morphological Noise Cleaning and Connectivity

- images perturbed by noise can be morphologically filtered to remove some noise

5.3.3 Openings, Holes, and Connectivity

- opening can create holes in a connected set that is being opened

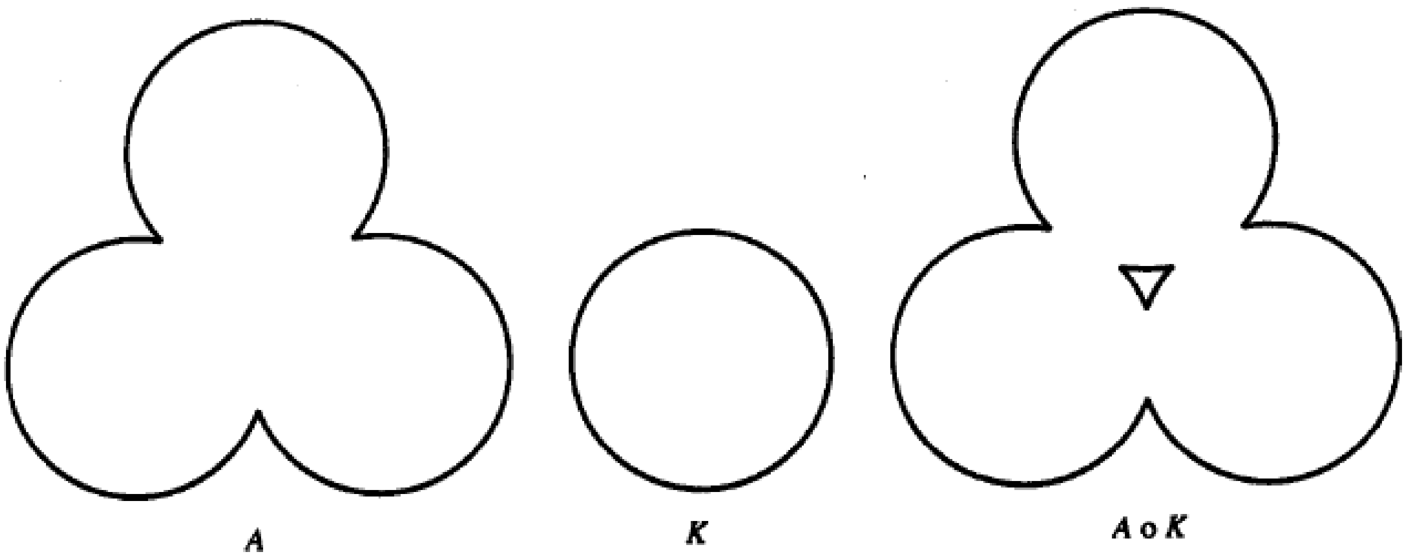


Figure 5.25 Introduction of holes by an opening. The connected set A has no holes. After being opened by a disk structuring element K, the opened set has a hole.

5.3.4 Conditional Dilation

- select connected components of image that have nonempty erosion
- conditional dilation $J \oplus|_I D$
- defined iteratively $J_0 = J$
- J are points in the regions we want to select

$$J_n = (J_{n-1} \oplus D) \cap I$$

- conditional dilation $J \oplus|_I D = J_m$
- where m is the smallest index $J_m = J_{m-1}$

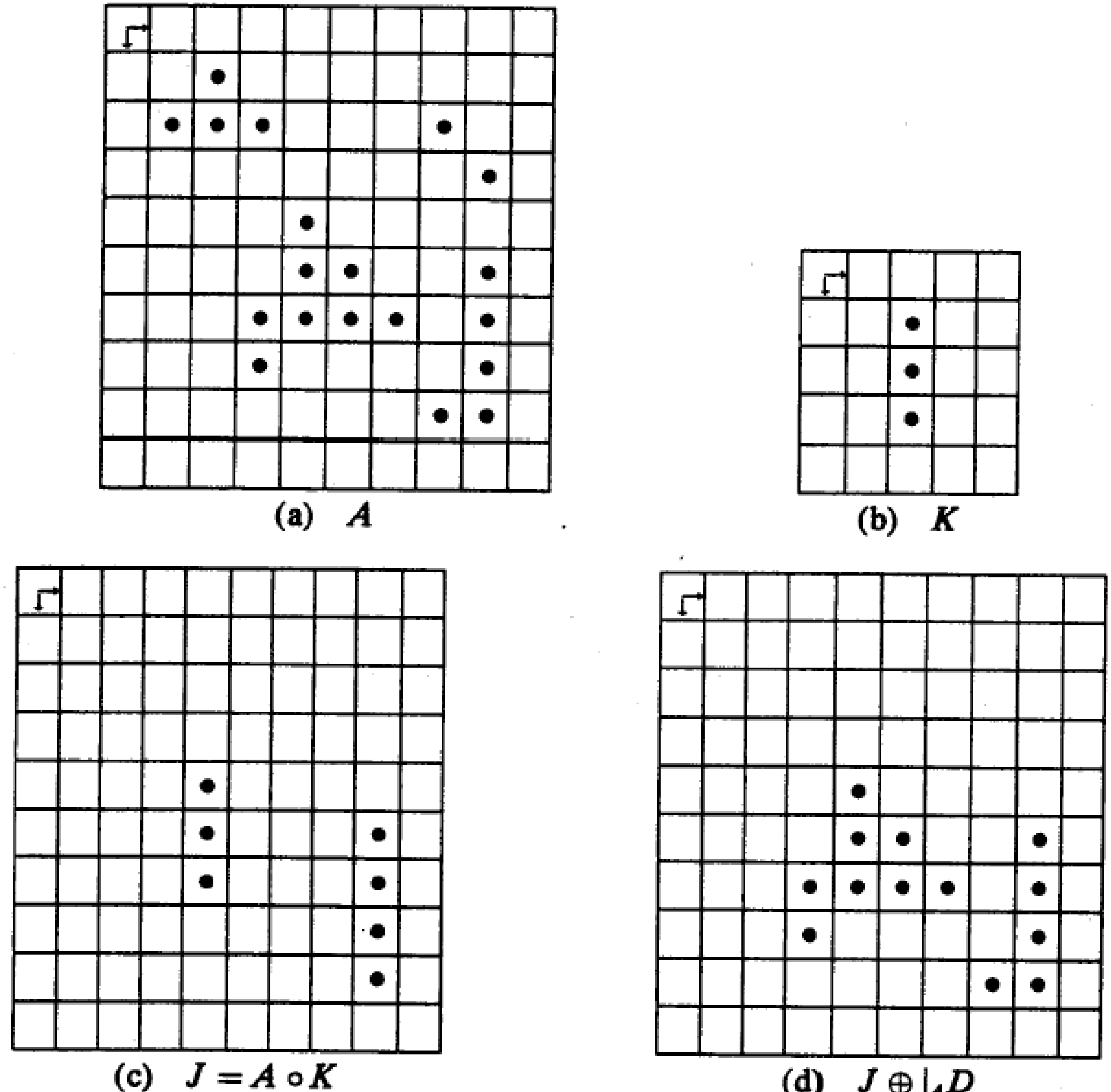


Figure 5.26 Use of the conditional dilation to select components having a particular property.

5.4 Generalized Openings and Closings

- generalized opening: any increasing, antiextensive, idempotent operation
- generalized closing: any increasing, extensive, idempotent operation

5.5 Gray Scale Morphology

- binary dilation, erosion, opening, closing naturally extended to gray scale
- extension: uses min or max operation
- gray scale dilation: surface of dilation of umbra
- gray scale dilation: maximum and a set of addition operations
- gray scale erosion: minimum and a set of subtraction operations

5.5.1 Gray Scale Dilation and Erosion

- top: top surface of *A*: denoted by

$$T[A] : F \to E:$$

$$T[A](x) = \max\{y \mid (x, y) \in A\}$$

- umbra of *f*: denoted by

$$U[f], U[f] \subseteq F \times E$$

$$U[f] = \{(x, y) \in F \times E \mid y \leq f(x)\}$$

5.5.1 Gray Scale Dilation and Erosion (cont')

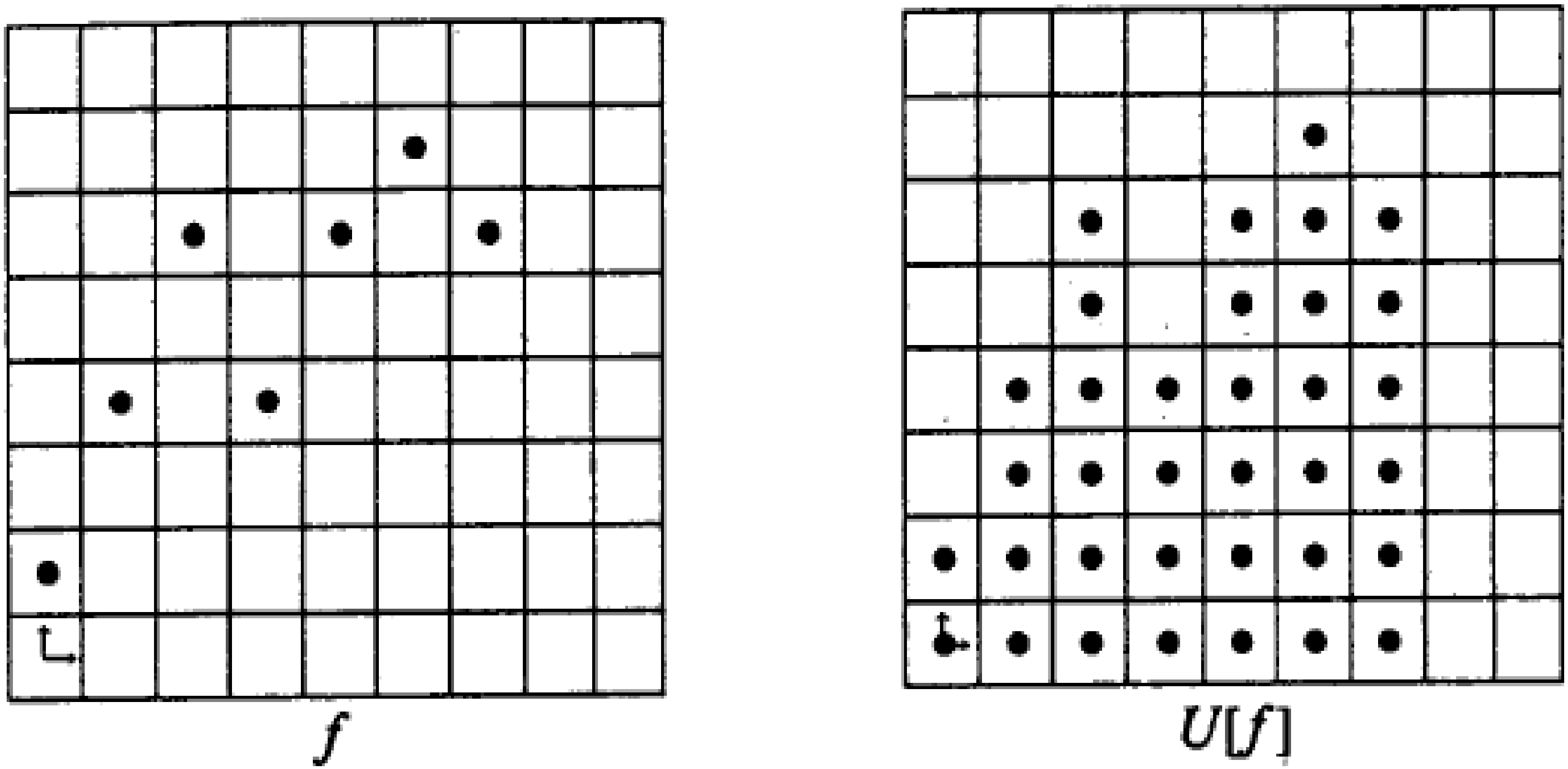


Figure 5.28 A function and its umbra.

5.5.1 Gray Scale Dilation and Erosion (cont')

- gray scale dilation: surface of dilation of umbras
- dilation of *f* by *k*: denoted by

$$f \oplus k = T\{U[f] \oplus U[k]\}$$

Figure 5.29 A small structuring element k and its umbra $U[k]$.

5.5.1 Gray Scale Dilation and Erosion (cont')

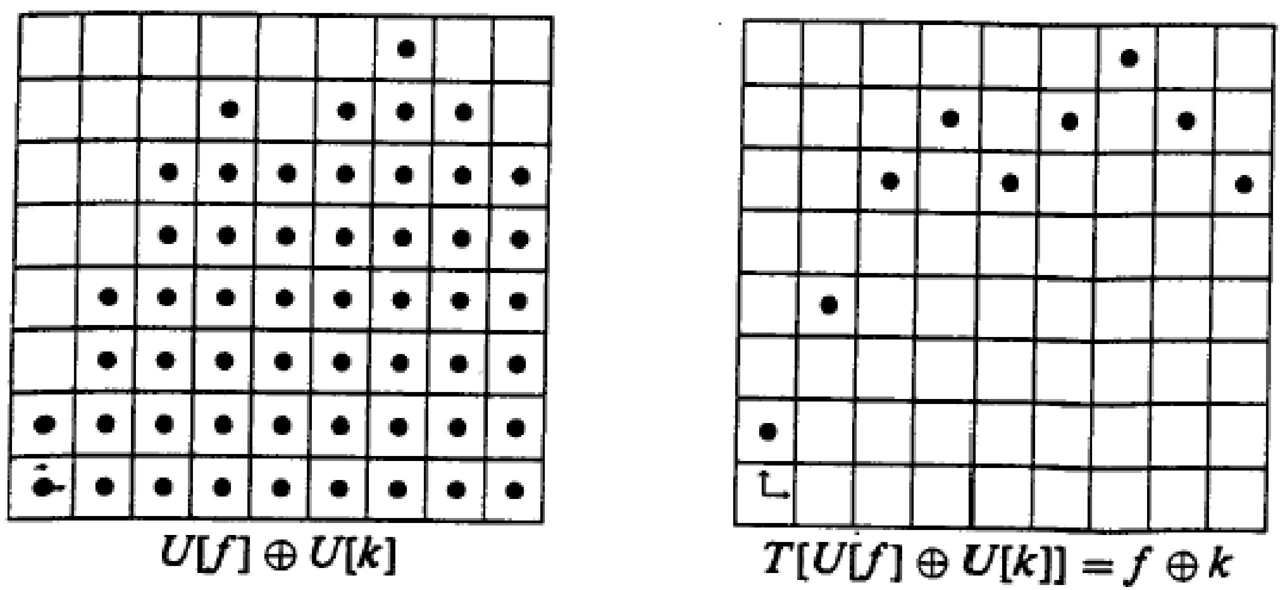


Figure 5.30 Gray scale dilation conceived by dilating the umbras and then taking the gray scale dilation to be the resulting top surface.

5.5.1 Gray Scale Dilation and Erosion (cont')

$f : F \rightarrow E$ and $k : K \rightarrow E$, then $f \oplus k : F \oplus K \rightarrow E$

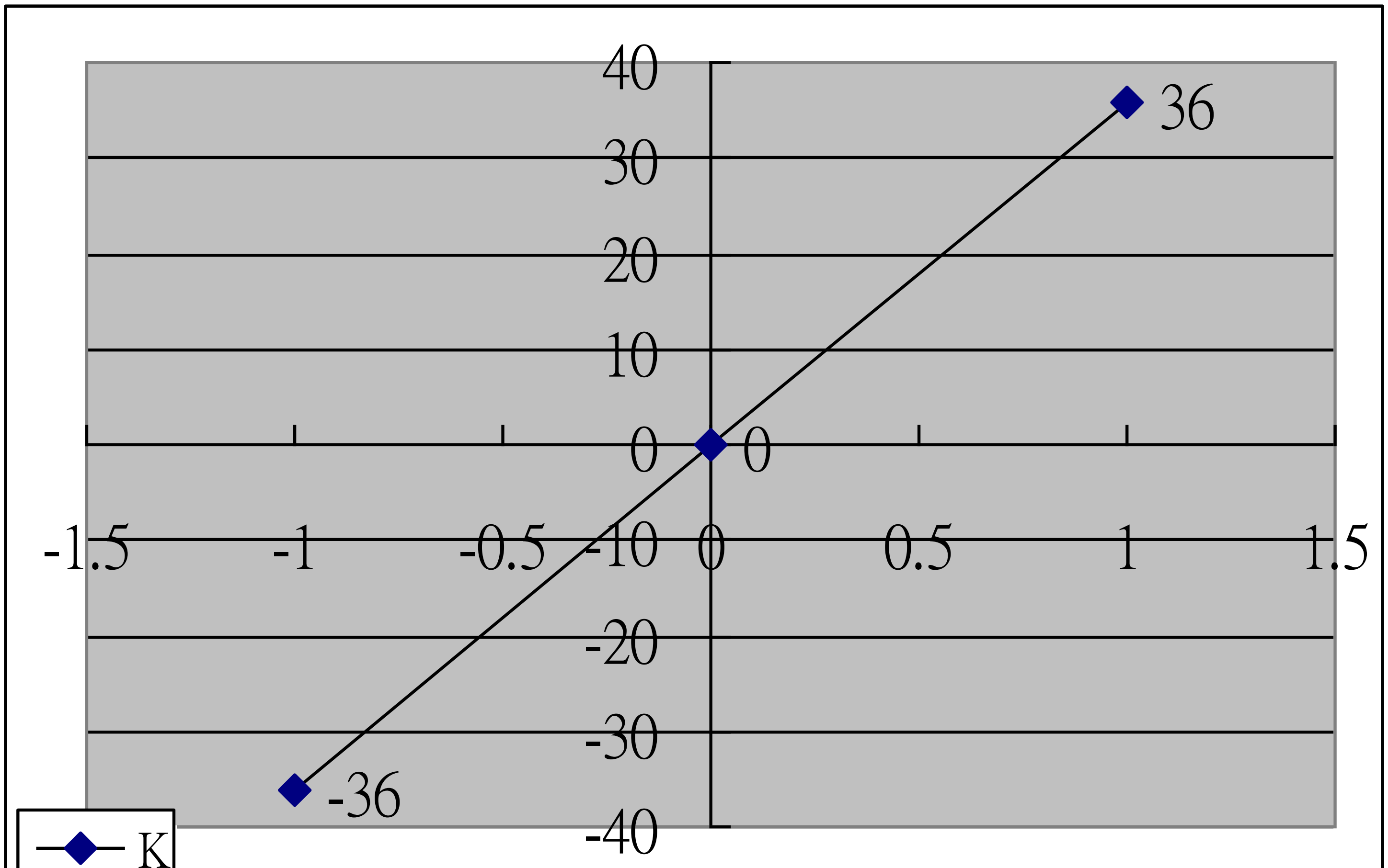
$$(f \oplus k)(x) = \max\{f(x - z) + k(z) | z \in K, x - z \in F\}$$

	K		
y	-1	0	1
$k(y)$	-36	0	36

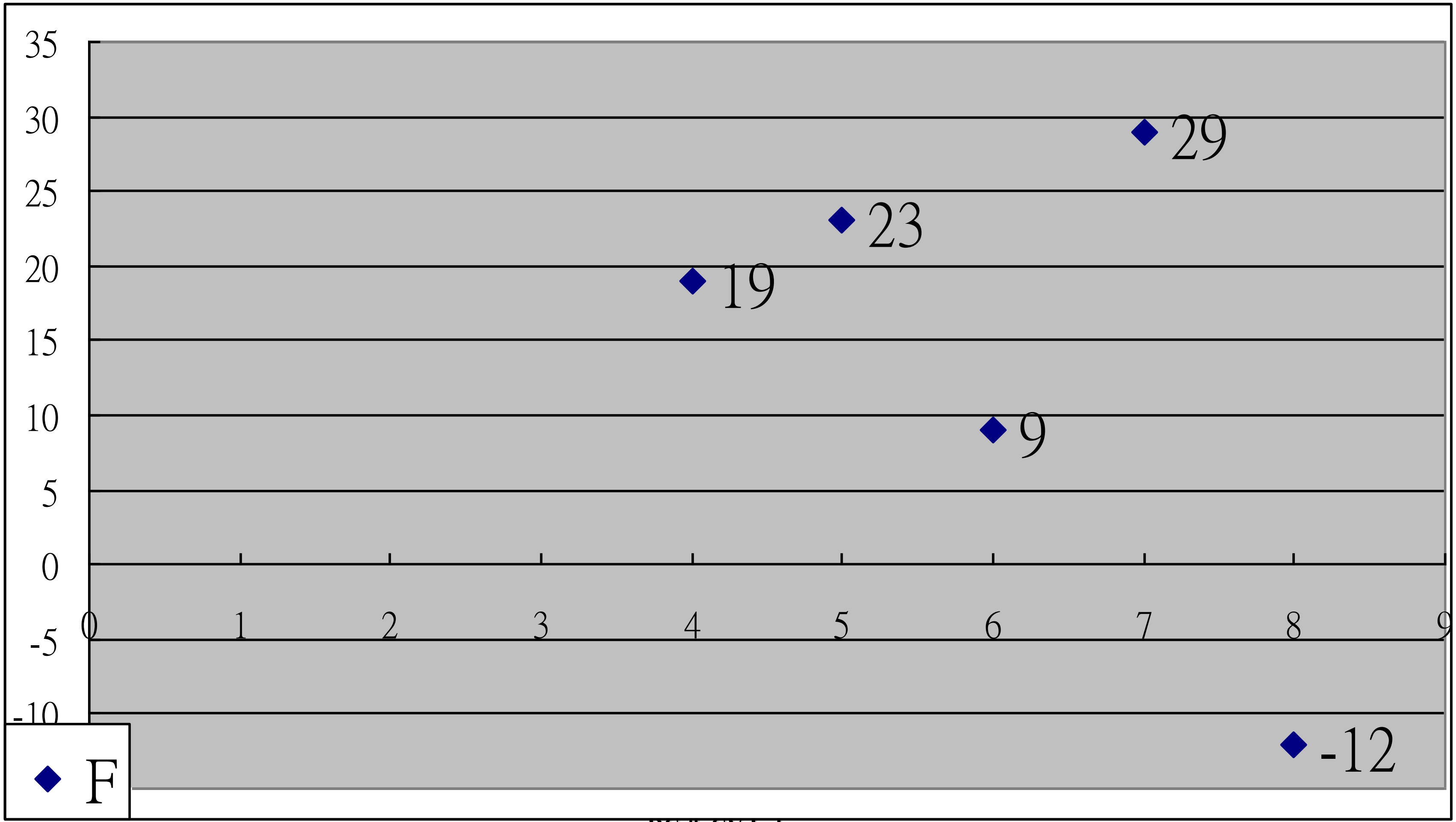
	F				
x	4	5	6	7	8
$f(x)$	19	23	9	29	-12

y	1	0	-1
$f(3-y)$	—	—	19
$k(y)$	36	0	-36
$f(3-y)+k(y)$	—	—	-17
$(f \oplus k)(3) = \max\limits_{\substack{y \in K \\ 3-y \in F}} f(3-y)+k(y)$			-17

5.5.1 Gray Scale Dilation and Erosion (cont')



5.5.1 Gray Scale Dilation and Erosion (cont')



5.5.1 Gray Scale Dilation and Erosion (cont')

y	1	0	-1
$f(4-y)$	—	19	23
$k(y)$	36	0	-36
$f(4-y)+k(y)$	—	19	-13
$(f \oplus k)(4) = \max\limits_{\substack{y \in K \\ 4-y \in F}} f(4-y)+k(y)$			19

y	1	0	-1
$f(5-y)$	19	23	9
$k(y)$	36	0	-36
$f(5-y)+k(y)$	55	23	-25
$(f \oplus k)(5) = \max\limits_{\substack{y \in K \\ 5-y \in F}} f(5-y)+k(y)$			55

Figure 5.31 Calculations for a gray scale dilation.

y	1	0	-1
$f(6-y)$	23	9	29
$k(y)$	36	0	-36
$f(6-y)+k(y)$	59	9	-7
$(f \oplus k)(6) = \max\limits_{\substack{y \in K \\ 6-y \in F}} f(6-y)+k(y)$			59

y	1	0	-1
$f(7-y)$	9	29	-12
$k(y)$	36	0	-36
$f(7-y)+k(y)$	45	29	-48
$(f \oplus k)(7) = \max\limits_{\substack{y \in K \\ 7-y \in F}} f(7-y)+k(y)$			45

y	1	0	-1
$f(8-y)$	29	-12	—
$k(y)$	36	0	-36
$f(8-y)+k(y)$	65	-12	—
$(f \oplus k)(8) = \max\limits_{\substack{y \in K \\ 8-y \in F}} f(8-y)+k(y)$			65

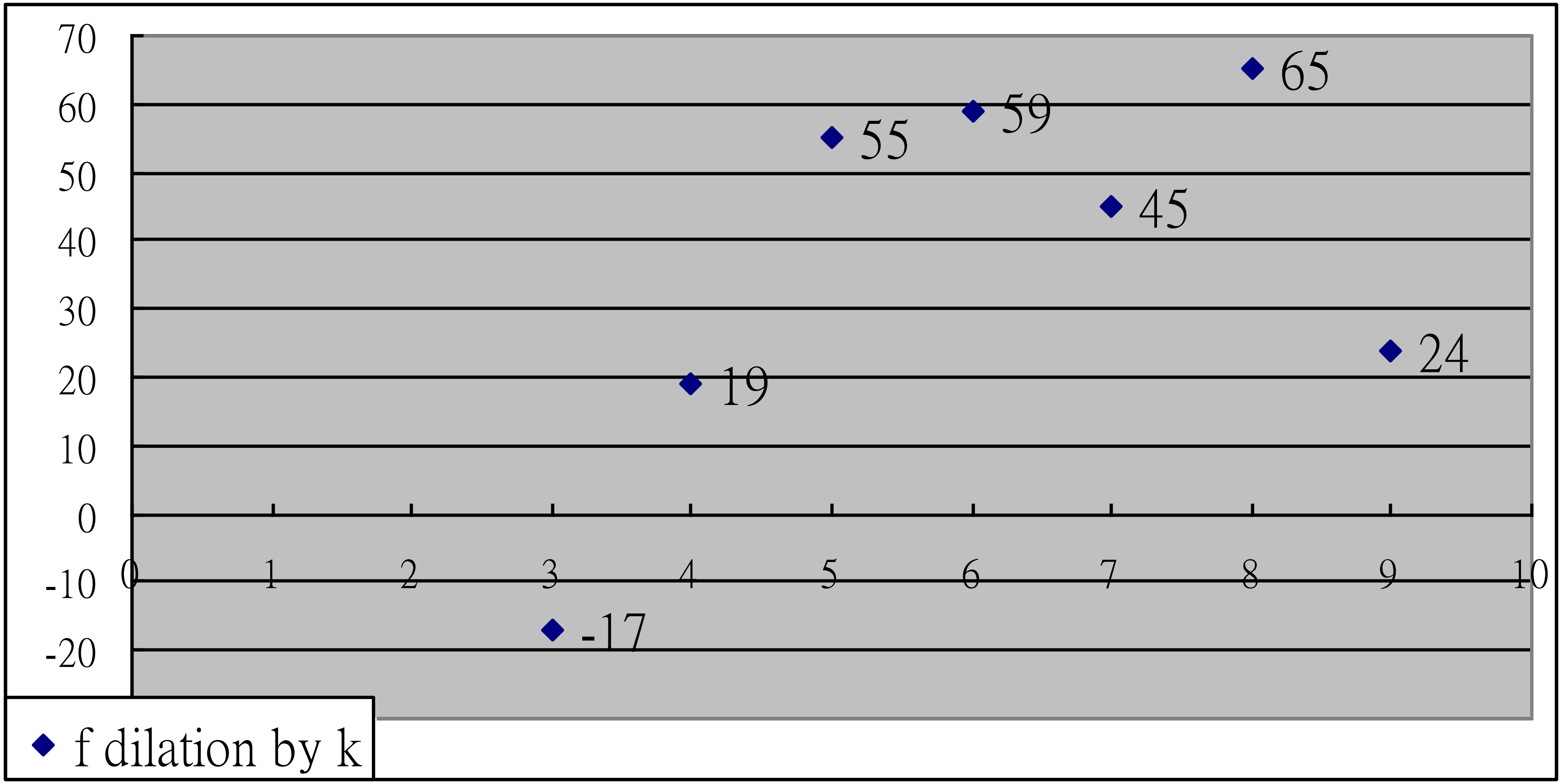
5.5.1 Gray Scale Dilation and Erosion (cont')

y	1	0	-1
$f(9-y)$	-12	—	—
$k(y)$	36	0	-36
$f(9-y)+k(y)$	24	—	—
$(f \oplus k)(9) = \max\limits_{\substack{y \in K \\ 9-y \in F}} f(9-y)+k(y)$			24

x	3	4	5	6	7	8	9
$f(x)$	—	19	23	9	29	-12	—
$(f \oplus k)(x)$	-17	19	55	59	45	65	24

Figure 5.31 *Continued.*

5.5.1 Gray Scale Dilation and Erosion (cont')



5.5.1 Gray Scale Dilation and Erosion (cont')

gray scale erosion: surface of binary erosions of one umbra by the other umbra

$$f \ominus k = T\{U[f] \ominus U[k]\}$$

5.5.1 Gray Scale Dilation and Erosion (cont')

=lena.im=

5.5.1 Gray Scale Dilation and Erosion (cont')

=lena.im.dil=

5.5.1 Gray Scale Dilation and Erosion (cont')

- Structuring Elements:

	*	*	*	
*	*	*	*	*
*	*	*	*	*
*	*	*	*	*
	*	*	*	

- Octagon
- Value = 0

5.5.1 Gray Scale Dilation and Erosion (cont')

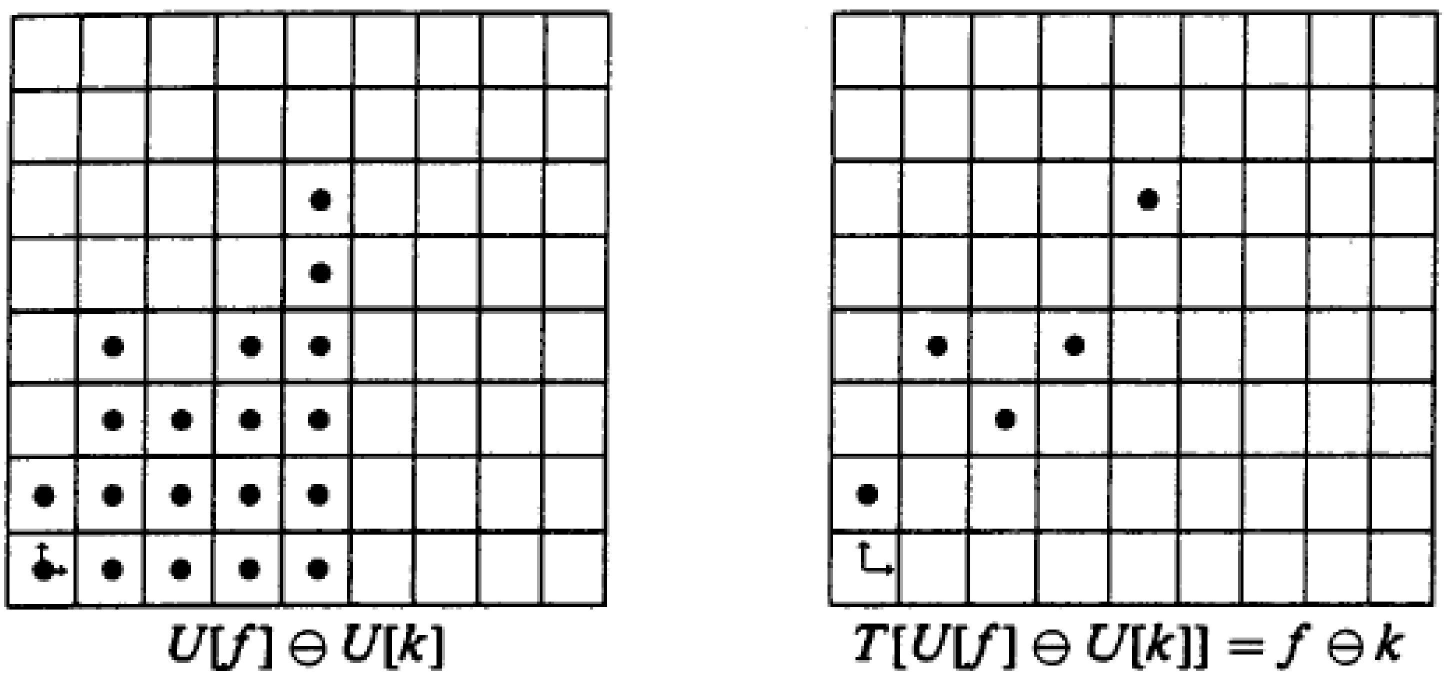


Figure 5.32 Gray scale erosion conceived by eroding the umbra of f by the umbra of k and then taking the gray scale erosion to be the resulting top.

5.5.1 Gray Scale Dilation and Erosion (cont')

$f : F \to E$ and $k : K \to E$, then $f \ominus k : F \ominus K \to E$

$$(f \ominus k)(x) = \min_{z \in K}\{f(x+z) - k(z)\}$$

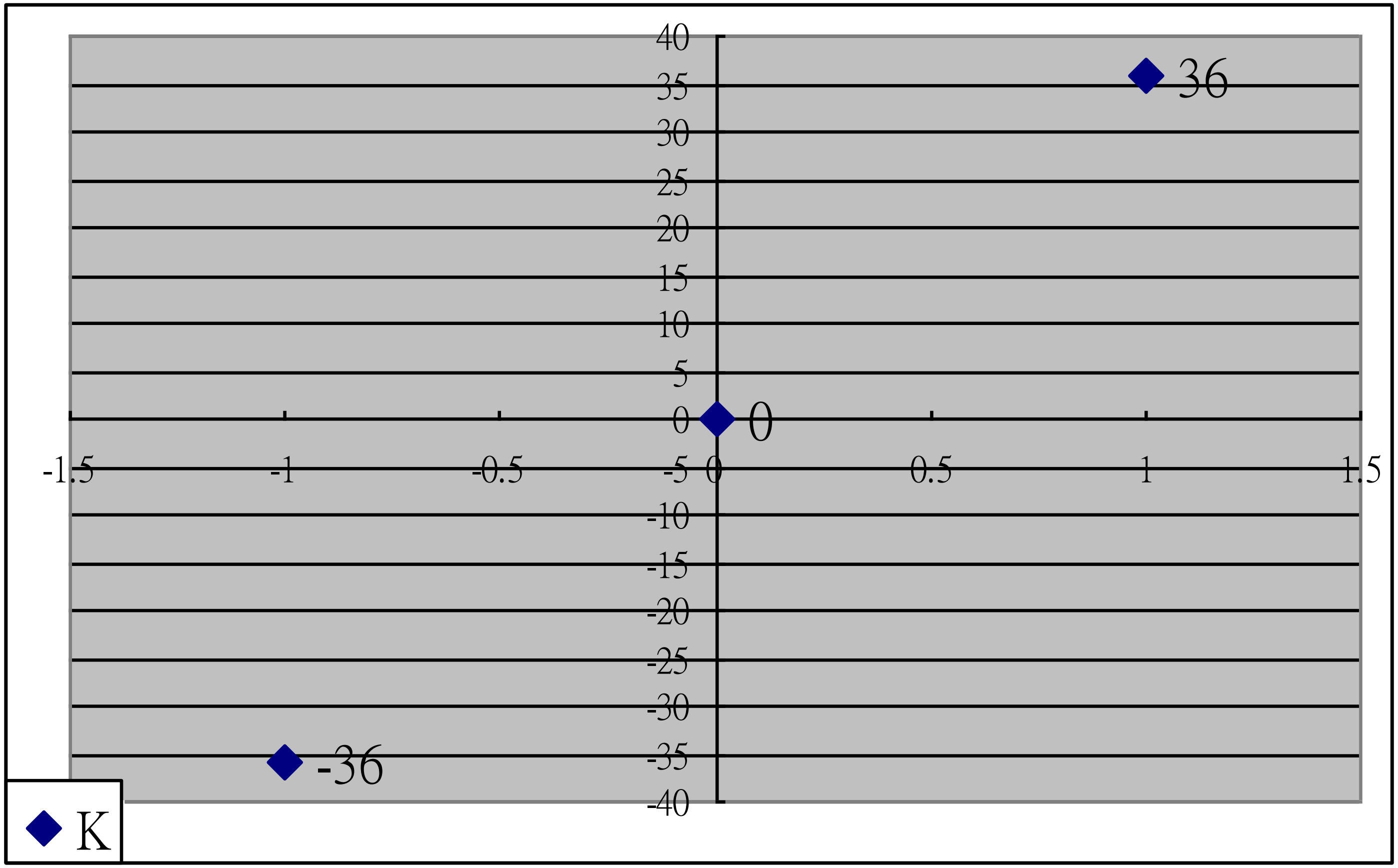
y	-1	0	1
$k(y)$	-36	0	36

x	3	4	5	6	7	8	9
$f(x)$	-12	31	55	59	45	7	-48

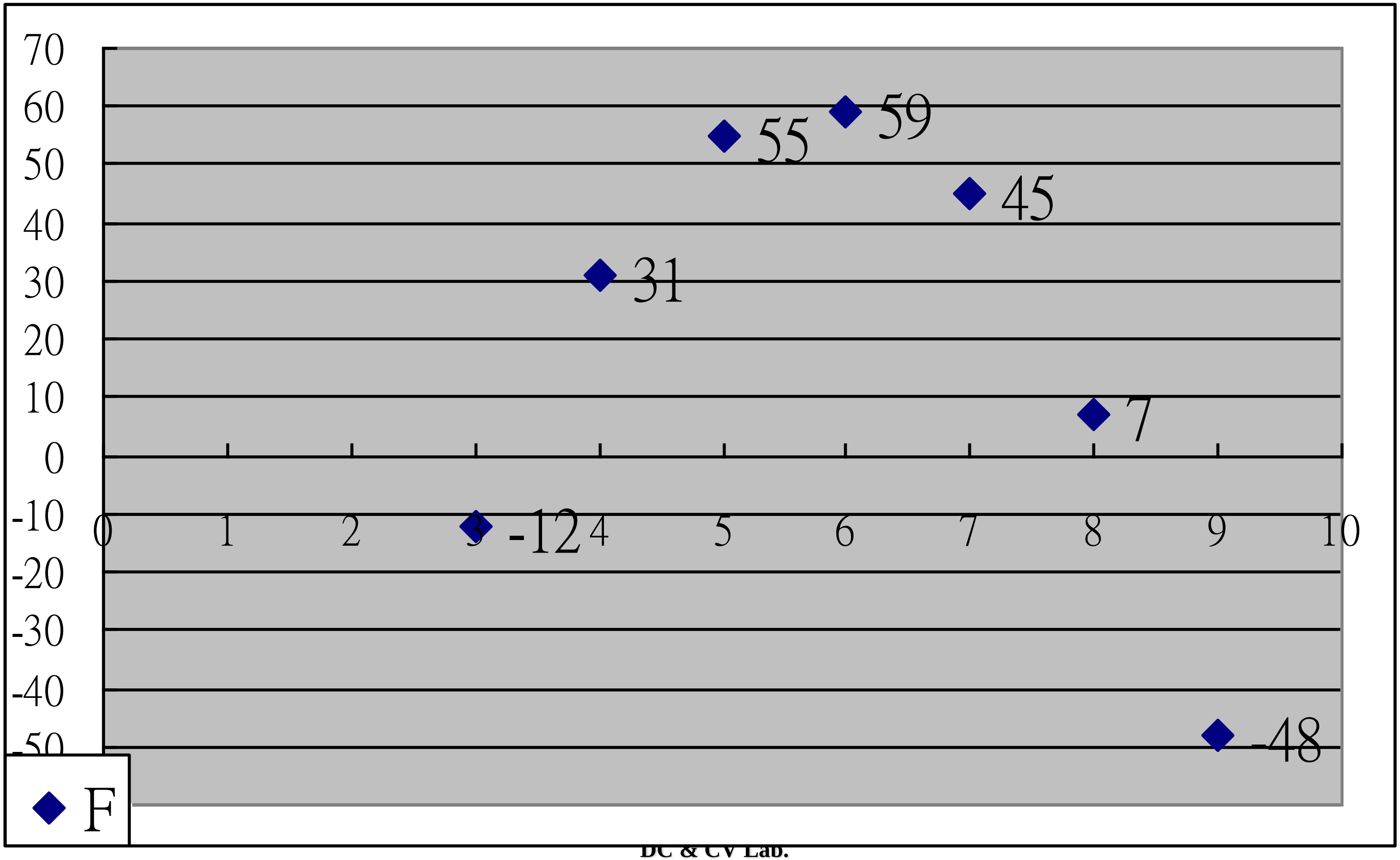
y	-1	0	1
$f(4+y)$	-12	31	55
$k(y)$	-36	0	36
$f(4+y) - k(y)$	24	31	19
$(f \ominus k)(4) = \min_{y \in K} f(4+y) - k(y)$			19

y	-1	0	1
$f(5+y)$	31	55	59
$k(y)$	-36	0	36
$f(5+y) - k(y)$	67	55	23
$(f \ominus k)(5) = \min_{y \in K} f(5+y) - k(y)$			23

5.5.1 Gray Scale Dilation and Erosion (cont')



5.5.1 Gray Scale Dilation and Erosion (cont')



y	-1	0	1
$f(6+y)$	55	59	45
$k(y)$	-36	0	36
$f(6+y)-k(y)$	91	59	9
$(f \ominus k)(6) = \min_{y \in K} f(6+y) - k(y)$			9

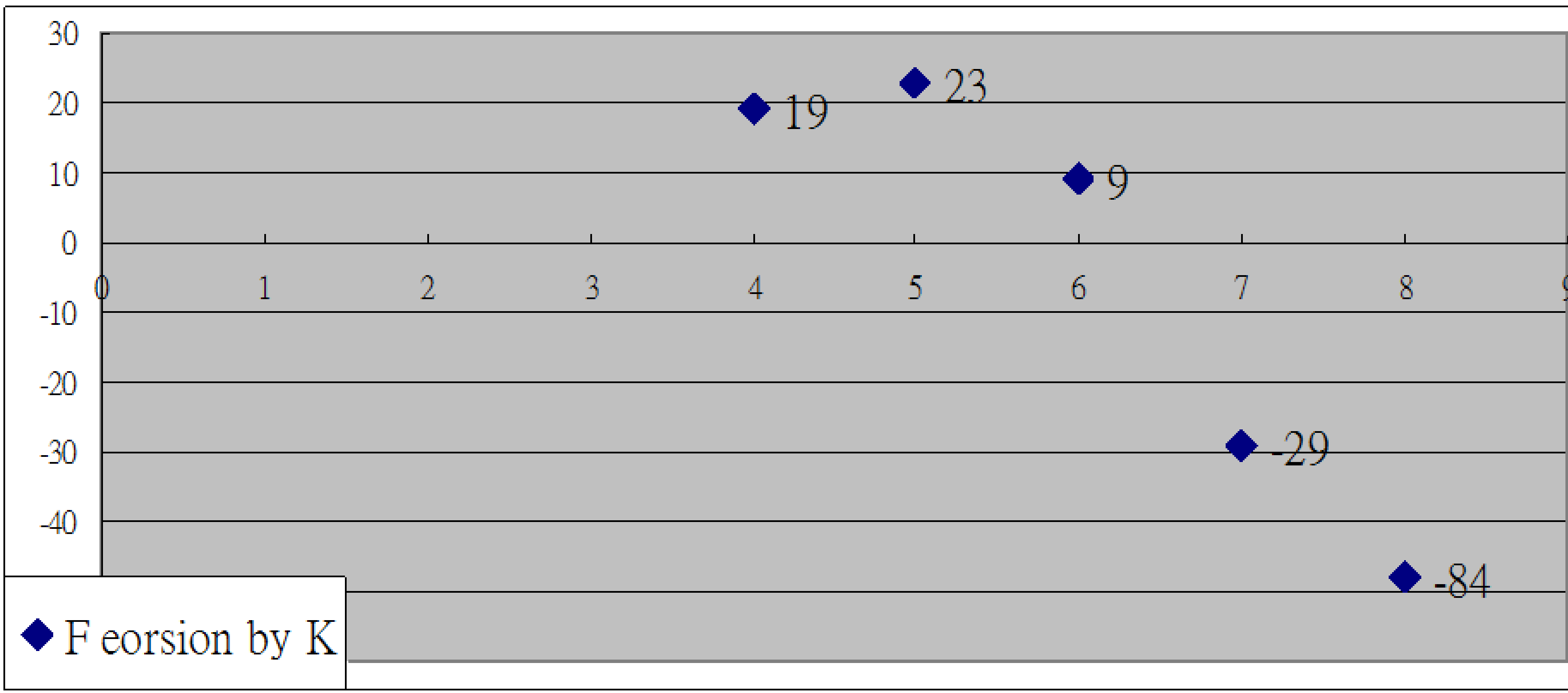
y	-1	0	1
$f(7+y)$	59	45	7
$k(y)$	-36	0	36
$f(7+y)-k(y)$	95	45	-29
$(f \ominus k)(7) = \min_{y \in K} f(7+y) - k(y)$			-29

y	-1	0	1
$f(8+y)$	45	7	-48
$k(y)$	-36	0	36
$f(8+y)-k(y)$	81	7	-84
$(f \ominus k)(8) = \min_{y \in K} f(8+y) - k(y)$			-84

x	3	4	5	6	7	8	9
$f(x)$	-12	31	55	59	45	7	-48
$(f \ominus k)(x)$	—	19	23	9	-29	-84	—

Figure 5.33 Calculations for a gray scale erosion.

5.5.1 Gray Scale Dilation and Erosion (cont')



5.5.1 Gray Scale Dilation and Erosion (cont')

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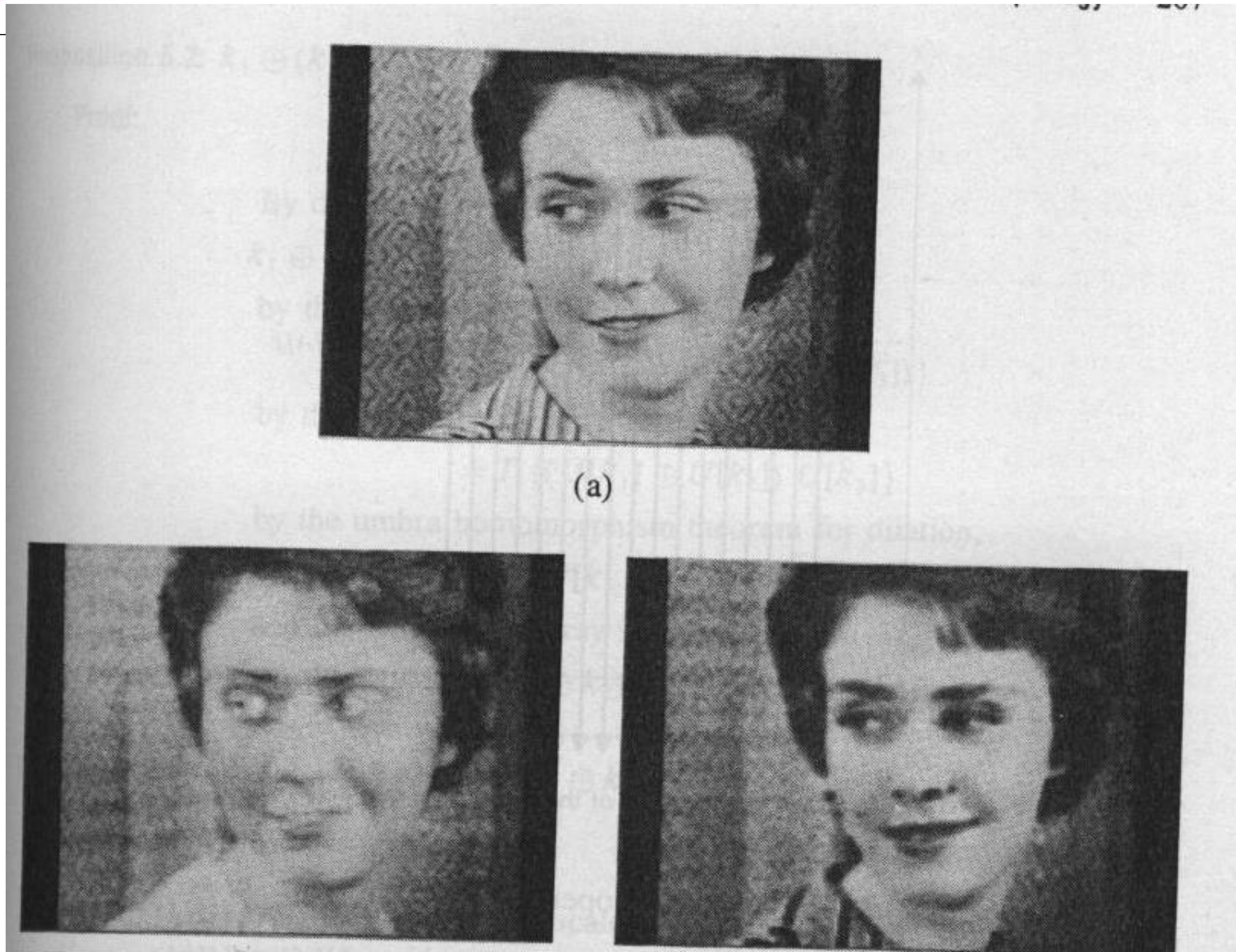
5.5.1 Gray Scale Dilation and Erosion (cont')

=lena.im.ero=

5.5.1 Gray Scale Dilation and Erosion (cont’)



5.5.1 Gray Scale Dilation and Erosion (cont')

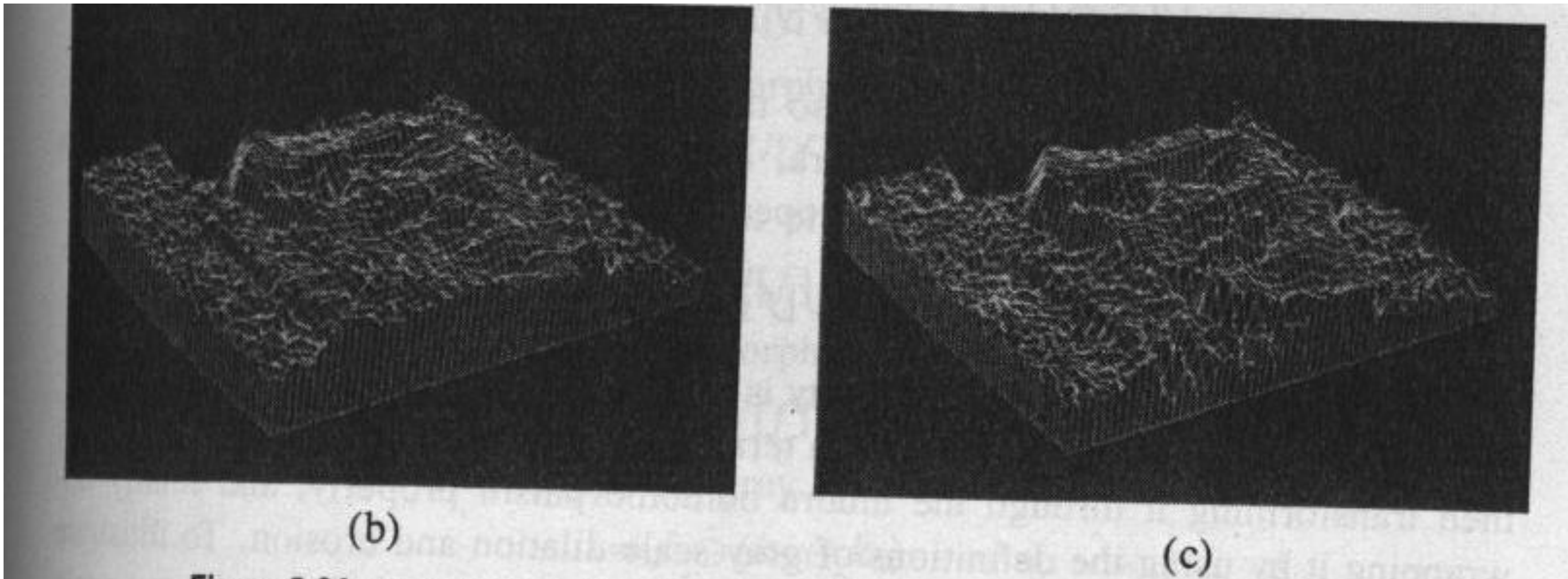


Figure 5.34 (a) A woman's face in an image form and in a perspective projection surface plot form. This image is morphologically processed with a paraboloid structuring element given by $6(8 - r^2 - c^2)$, $-2 \le r \le 2$, $-2 \le c \le 2$. (b) Erosion of the woman's face in image form and perspective projection surface plot form. (c) Dilation of her face in image form and perspective projection surface plot form.

5.5.2 Umbra Homomorphism Theorems

- surface and umbra operations: inverses of each other, in a certain sense
- surface operation: left inverse of umbra operation $T\{U[f]\} = f$

Theorem 5.3: Umbra Homomorphism Theorem

Let $F, K \subseteq E^{N-1}$ and let $f : F \to E$ and $k : K \to E$. Then

1. $U[f \oplus k] = U[f] \oplus U[k]$
2. $U[f \ominus k] = U[f] \ominus U[k]$

5.5.2 Umbra Homomorphism Theorems

- Proposition 5.1 $f \oplus k = k \oplus f$
- Proposition 5.2 $k_1 \oplus (k_2 \oplus k_3) = (k_1 \oplus k_2) \oplus k_3$
- Proposition 5.3 $(f \ominus k_1) \ominus k_2 = f \ominus (k_1 \oplus k_2)$

5.5.3 Gray Scale Opening and Closing

- gray scale opening of *f* by kernel *k* denoted by $f \circ k$

$$f \circ k = (f \ominus k) \oplus k$$

- gray scale closing of *f* by kernel *k* denoted by $f \bullet k$

$$f \bullet k = (f \oplus k) \ominus k$$

5.5.3 Gray Scale Opening and Closing (cont')

- =lena.im.open=

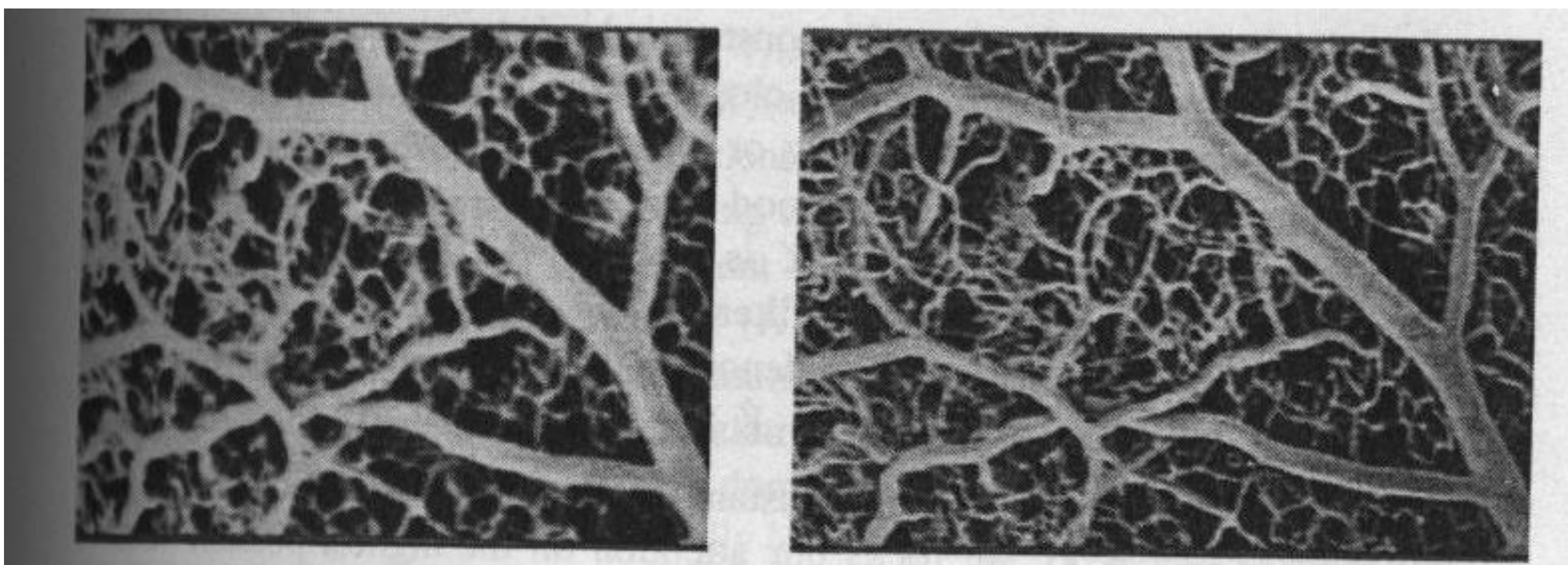
5.5.3 Gray Scale Opening and Closing (cont')

- =lena.im.close=

5.5.3 Gray Scale Opening and Closing (cont')

- duality of gray scale dilation, erosion ➔ duality of opening, closing

$$-(f \circ k)(x) = [(-f) \bullet \check{k}](x)$$



(a) (b)

5.5.3 Gray Scale Opening and Closing (cont’)

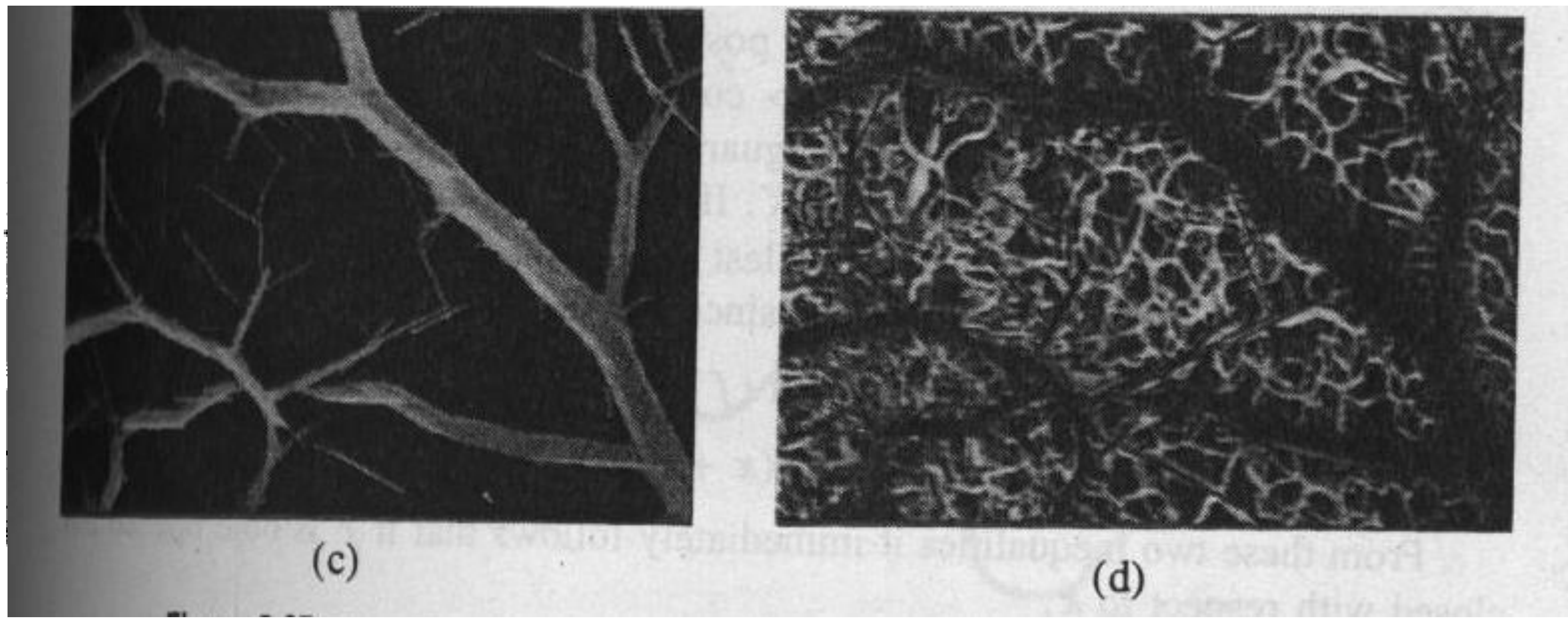


Figure 5.37 Example of how the generalized opening can be used. (a) Original image f; (b) sharpened image g; (c) $h = k \wedge m$, where $k = \max\{g \circ l_1, \ldots, g \circ l_8\}$, $m = k > t$, and $l_1, \ldots, l_8$ are line structuring elements in direction 0°, 27°, 45°, 63°, 90°, 117°, 135°, 153°; and (d) $g - h$.

5.6 Openings, Closings, and Medians

- median filter: most common nonlinear noise-smoothing filter
- median filter: for each pixel, the new value is the median of a window
- median filter: robust to outlier pixel values, leaves sharp edges
- median root images: images remain unchanged after median filter

Original Image

Median Root Image

5.7 Bounding Second Derivatives

- opening or closing a gray scale image simplifies the image complexity

5.8 Distance Transform and Recursive Morphology

$$J(r,c) = \begin{cases} \min\{J(r,c-1), J(r-1,c)\} + 1, & \text{if } I(r,c) = 1 \\ 0, & \text{if } I(r,c) = 0 \end{cases}$$

$$D(r,c) = \min\{J(r,c), D(r,c+1)+1, D(r+1,c)+1\}$$

5.8 Distance Transform and Recursive Morphology (cont')

- Fig 5.39 (b) fire burns from outside but burns only down-ward and right-ward
-

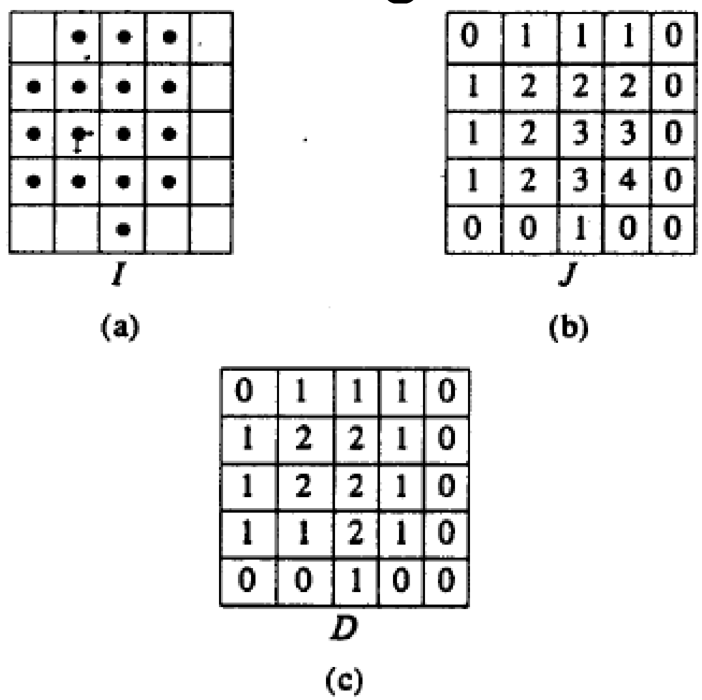


Figure 5.39 (a) An example binary input image; (b) the forward scan output image J produced by a recursive morphological operation with the structuring element $F = \{(0,-1),(-1,0)\}$; and (c) the city-block distance transform of image I produced by a recursive morphological operation on image J with the structuring element $B = \{(0,0),(0,1),(1,0)\}$.

5.9 Generalized Distance Transform

$$d[A,K](x) = \max\{m \mid x \in A \underset{m-1}{\ominus} K\}$$

$$= \max\{m \mid x \in A \ominus (\underset{m-1}{\oplus} K)\}$$

$$= \max\{m \mid (\underset{m-1}{\oplus} K)_x \subseteq A\}$$

5.10 Medial Axis

- medial axis transform: medial axis with distance function

5.10.1 Medial Axis and Morphological Skeleton

- morphological skeleton of a set *A* by kernel *K*, where $A \ominus_0 K = A$

$S_0, \ldots, S_N$, where $S_n = A \ominus_n K - (A \ominus_n K) \circ K$

skeleton of A given by $\bigcup_{n=0}^{N} S_n$

5.10.1 Medial Axis and Morphological Skeleton (cont’)

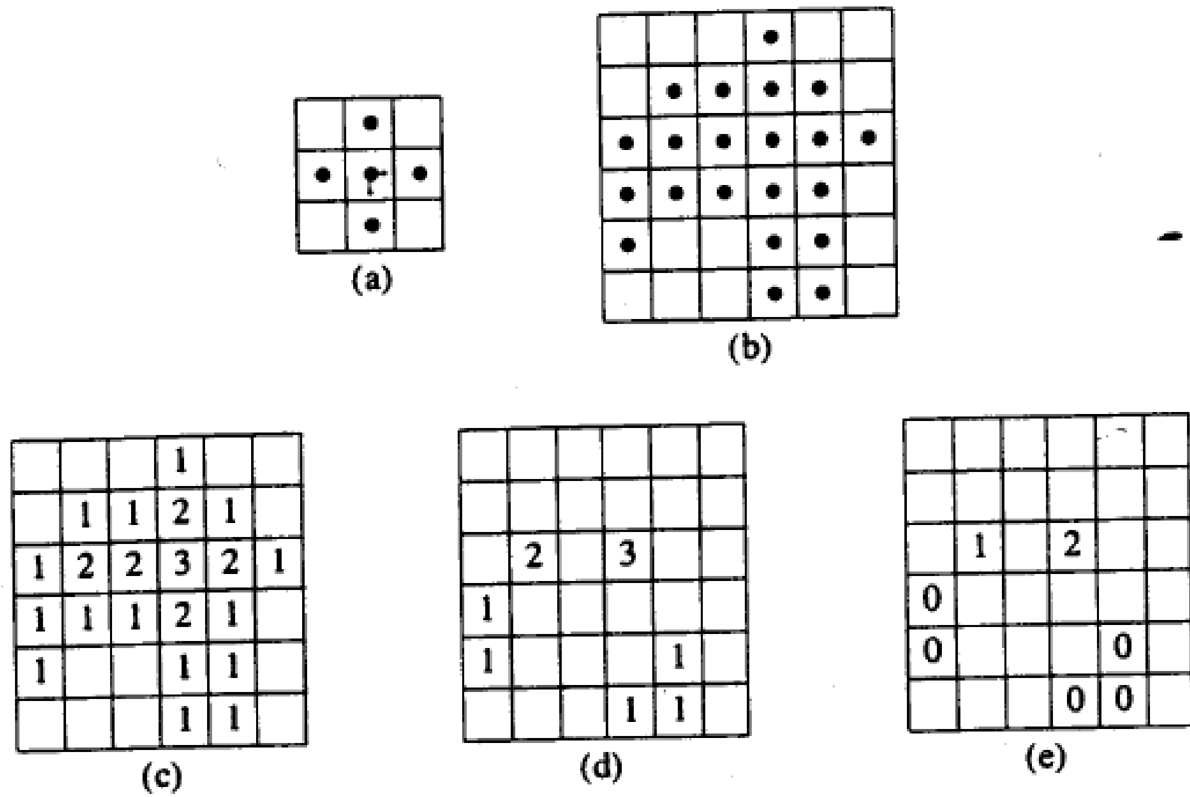
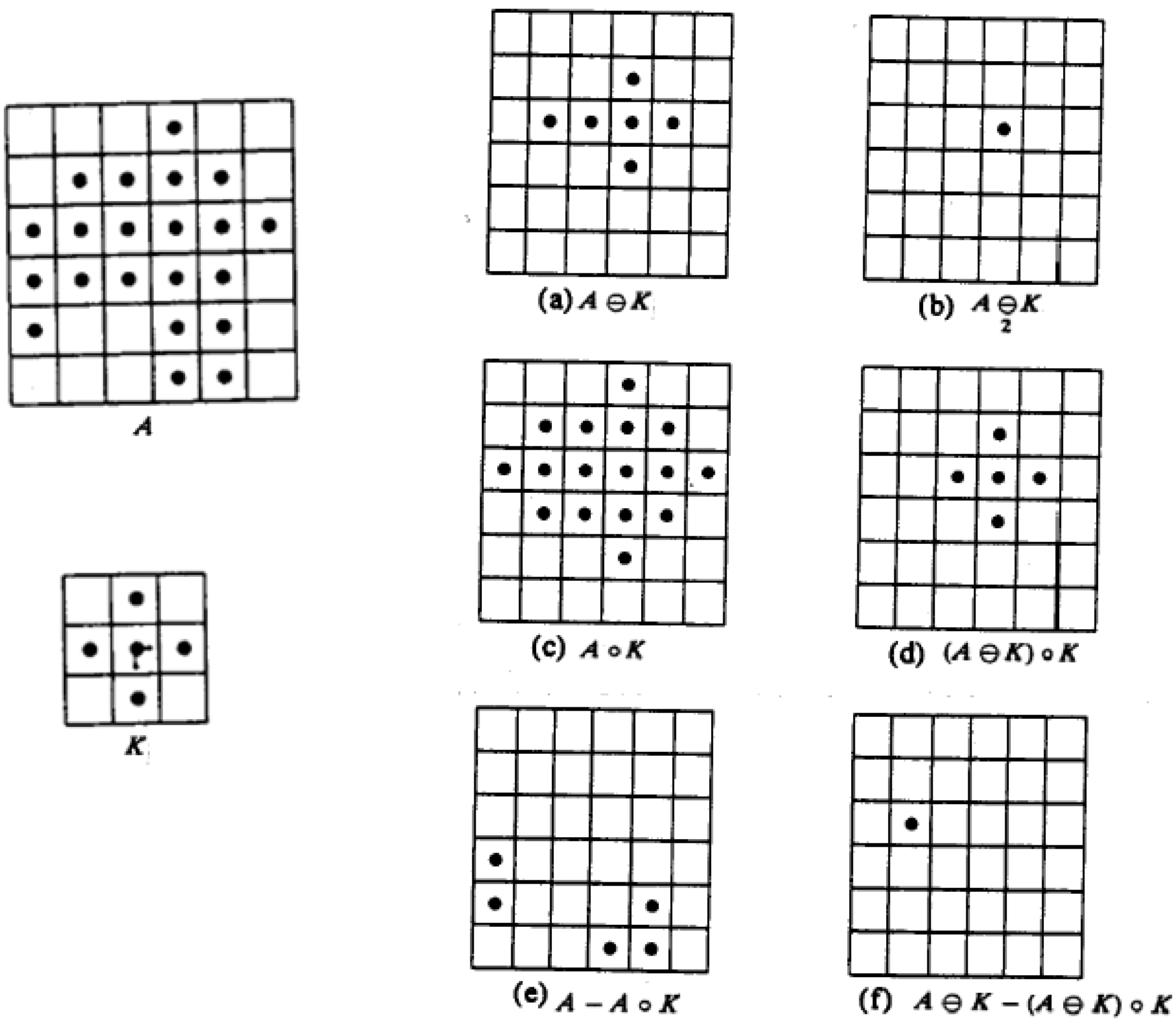
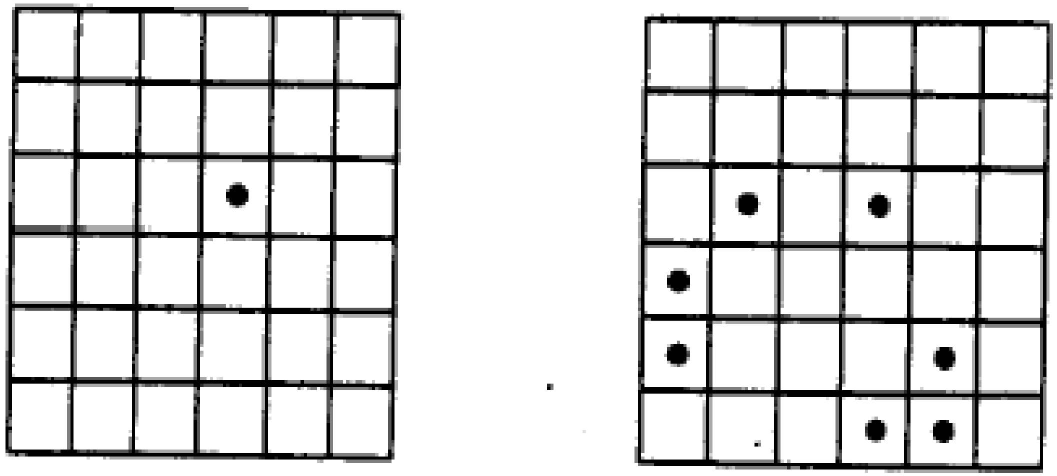


Figure 5.40 Generation of the medial axis from the relative extrema of the distance transform. (a) Structuring element K; (b) A; (c) $d[A,K](x)$ for $x \in A$; (d) $Ext[A,K]$ and associated $d[A,K](x)$ for $x \in Ext[A,K]$; and (e) $MA[A,K]$ and associated medial axis function. Only those pixels with values shown are part of the medial axis.

5.10.1 Medial Axis and Morphological Skeleton (cont')



5.10.1 Medial Axis and Morphological Skeleton (cont')



(g) (h)

Figure 5.41 Calculation of the morphological skeleton for the set A with respect to structuring element K. (a) $A \ominus K$; (b) $A \underset{2}{\ominus} K$; (c) $A \circ K$; (d) $(A \ominus K) \circ K$; (e) $S_0 = A - A \circ K$; (f) $S_1 = A \ominus K - (A \ominus K) \circ K$; (g) $S_2 = A \underset{2}{\ominus} K - (A \underset{2}{\ominus} K) \circ K$; and (h) $S = S_0 \cup S_1 \cup S_2$. Both A and K are shown in Fig. 5.40.

5.11 Morphological Sampling Theorem

- Before sets are sampled for morphological processing, they must be morphologically simplified by an opening or a closing.
- Such sampled sets can be reconstructed in two ways: by either a closing or a dilation.

5.12 Summary

- morphological operations: shape extraction, noise cleaning, thickening
- morphological operations: thinning, skeletonizing

Homework

- Write programs which do binary morphological dilation, erosion, opening, closing, and hit-and-miss transform on a binary image (Due Oct. 29)
- Write programs which do gray-scale morphological dilation, erosion, opening, and closing on a gray-scale image (Due Nov. 5)